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and

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LONDON: PRINTED BY
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AND PARLIAMENT STREET

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MENSURATION



BY

CHARLES W. MERRIFIELD, F.R.S.

BARRISTER - AT - LAW,

Principal of the Royal School of Naval Architecture and Marine Engineering,
Honorary Secretary of the Institution of Naval Architects, and
Late an Examiner in the Department of Public Education.

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1873.

PREFACE.

THIS TREATISE on Technical Arithmetic and Mensuration has been written as one of a series of Text Books on Science. It is not meant to be the first book of a child wholly uninstructed in the elements of arithmetic. No one book could be fitted for such teaching and for advanced instruction also. The task to which the author has addressed himself is that of giving to the elementary rules the precision and illustration which they need for the further pursuit of the subject, and to the higher rules that gradual induction which is a more effective instrument of teaching than a strict logical arrangement. He trusts that his explanations will not be found wanting in accuracy of statement; but his experience has led him to believe that there is not much practical connection between successful teaching and logical sequence. The province of logic is to test ideas, not to impart them. The previous knowledge, which the student is assumed to possess, need not, however, be great in quantity or of high quality.

Care has been taken not to introduce anything in

the way of mathematical invention or discovery. All that this book contains is thoroughly well known and established. If there be any originality at all, it is in the selection and arrangement of materials.

It may be of service to teachers to point out what the author considers the distinctive features of the book. The first of these is the division of Fractions into two marked periods: the rule of greatest common measure being postponed until the elementary rules of fractions have been explained. This appears to the author to be less artificial, and easier to the learner, if less logical. The second is the finding the cube root by substitution in a simple and easily remembered formula (well known to algebraists) in preference to giving the rule for extracting the root, a rule which can hardly be understood or remembered by any person unacquainted with algebra. The third feature is a bold and somewhat hazardous experiment, the introduction of a chapter on MECHANICAL WORK. Most teachers are aware that this subject is generally deferred until too late, and that few but advanced students ever get any knowledge of it at all. It is not mere arithmetic (neither is mensuration) but it is assuredly not out of place in a book which is to be read by mechanics. No person who understands that chapter is at all likely to entertain the fallacy of supposing that a machine can produce perpetual motion. The chief danger

lies in the possibility of persons making an unintelligent use of the formulæ; but the chance of abuse seems scarcely a reason for suppressing useful information.

The treatise on Mensuration is intended as an arithmetical supplement to geometry, and not as a substitute for geometrical knowledge. The author has endeavoured to bring into distinctness the leading principles on which the application of arithmetic to geometry depends, such as the separation of size and form, the principle of similarity, the principle of deformation, and the coincidence of inferior and superior limits. These are not dealt with in elementary works, and not systematically grouped in higher text books, although all arithmetical geometry depends upon them.

It has been thought useful to add a selection of Examination Papers actually set at different examinations. These will show how much a student is usually expected to do in a given time. The examples contained in them will also afford a change from those set by the author in the course of the work.

While the author makes no claim to originality, he thinks it proper to mention, that, with the exception of the examination papers, the whole text has been written by himself, and that most of the examples have been set by him. He has not intentionally

copied or extracted anything, except the Units and Tables of Weights and Measures and their comparisons, and a very few examples. In every other respect the work is strictly his own.

ROYAL SCHOOL OF NAVAL ARCHITECTURE,
SOUTH KENSINGTON:
October 1871.

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ARITHMETIC.

MENSURATION.

TECHNICAL

ARITHMETIC and MENSURATION

CHAPTER I.

NOTATION AND NUMERATION.

THE MAIN OBJECTS of Arithmetic are—counting and recording the result. The first operation is called numeration, and the second notation. The requirements of each have affected the practice of the other.

The first idea is to make a mark for every object among those which we wish to count. This kind of tally is indeed a numerical record of what we have observed, but it fails to give us the evident comparison which we want in all our dealings. There is no arithmetic here. The next step is to accustom ourselves thoroughly to the comparison of small groups of these individual marks, which we call *units*, so as to arrive at the distinction between two, three, four, five, and so forth. It is very soon found that our power of inventing names in this direction comes to an end, and we have been driven into improved arrangements.

The fingers were the first mechanical aid in counting, but these were not the only possible mode of doing this, nor the best permanent record. Still their total number—ten—is the foundation of our arithmetic. What we usually do amounts to this: we count right up to ten, and at the tenth we make a mark; we then begin to count again from one to ten, and

then we make another mark, getting two tens; we then begin again, obtaining in succession three tens, four tens, five tens, and so on. Well, if we have very many to count, we find that the number of marks, each of which represents ten, itself becomes very high, and in its turn requires a shortened method of counting and recording. For this purpose, whenever we get ten tens we make one mark in a different place, and we do the same for every ten tens. By this plan, what we commonly know as four hundred and twenty-seven comes to be written as—

*	*	*
*	*	*
*		*
*		*
		*
		*
		*

The marks in the right-hand column represent one for each object counted; each of those in the next column is made to do duty for ten of the former ones, and each of those in the left-hand column for ten of those in the middle column.

We are not yet entitled to say what relation connects the left and right hand columns further than that each mark in the left-hand column stands for ten tens, whatever that may be.

Two more steps now need to be noticed; the first, which was the earliest, was to substitute a separate mark for each group of marks less than ten; this gave us the nine *digits*, as they are called, which we write as—

1, 2, 3, 4, 5, 6, 7, 8, 9

The second step, which appears to have been known in India long ago, but to have been brought into Europe by the Moors and Arabs, was to distinguish the tens and the tens of tens by the mere position of the figures. This is a point which we must explain at some length, because it presents a little difficulty, the right understanding of which

will save us much confusion, and indeed will remove all difficulty when we come by-and-by to what are called Decimal Fractions.

So long as we have units, tens, and tens of tens, each of them in some specific number, as 427, there is no difficulty about the matter; but there *is* a difficulty if there be a whole number of tens in one way; or in another, if there be no tens over from the tens of tens. For example, suppose there are four tens of tens and seven units; if we write this down as 47, it might be taken to mean four tens and seven units. In the same way, if we had four tens of tens, and seven tens without any spare units, and were to write them down as 47, we should be in the like difficulty. Now this difficulty arises simply because we have as yet devised no means of making the figures keep their place when there happens to be a blank amongst them. To complete our notation, therefore, we want a special mark to indicate a blank: this mark by itself must have no meaning. The duty it has to perform is simply to keep a place. In our practice we use o for this purpose, and we call it nought or zero. In this way we have no difficulty in distinguishing four hundred and seventy from four hundred and seven, or forty-seven. We write them as—

470 407 and 47

What we have written is merely an illustration of our notation and numeration. In detail it is as in the following example:—

		Totals
2 units	2	
5 tens	50	52
1 hundred . . .	100	152
7 thousands . . .	7,000	7,152
9 tens of thousands. .	90,000	97,152
3 hundreds of thousands.	300,000	397,152
5 millions . . .	5,000,000	5,397,152

Thus the whole amount stated above would be written as

5,397,152, and would be read as five *millions*, three hundred and ninety-seven *thousand*, one hundred and fifty-two (*units*). For the convenience both of counting the figures and reading them, we divide them into groups of three. The reason for choosing groups of *three* is because we can read them off in thousands, and in thousands of thousands or millions, and so on; for that is the way in which it is usual to count high numbers.

The actual scale is as follows:—

Units . . a single one being written as		1
Tens	"	10
Hundreds	"	100
Thousands	"	1,000
Tens of thousands	"	10,000
Hundreds of thousands	"	100,000
Millions	"	1,000,000
Tens of millions	"	10,000,000
Hundreds of millions	"	100,000,000
Thousands of millions	"	1,000,000,000
Tens of thousands of millions	"	10,000,000.000
Hundreds of thousands of millions	"	100,000,000,000
Billions	"	1,000,000,000,000

It is worth while to remark that as regards *billions* there is a difference between the French and English practice; in French a billion (or *milliard*) is one thousand millions, in English a billion is a million millions, and accordingly the word is seldom used in our language; for such large numbers are rarely of any practical use.

The old books use a scale of numbers of this kind:—

A million of millions is a billion,
A million of billions is a trillion,

and so forth; but these names are never used in practice, and can hardly be said to belong to the language of arithmetic or to English speech.

As another example we may take the following number—

655,991,810,819,852

This is read as—

Six hundred and fifty-five billions, nine hundred and ninety-one thousand eight hundred and ten millions, eight hundred and nineteen thousand, eight hundred and fifty-two.

The student will do well to practise himself both in reading sets of figures aloud by their full names, and also in writing down the figures corresponding to long stated numbers.

When figures have to be copied or written from dictation, it is best to give the proper name of the whole number first, and then to call over the figures separately. The latter is the more important for correctness, especially when the numbers run heavy. For this purpose no *teens* or '*tys*' should be used, but the number should be called over (using the last example) as follows :—

Six five five, nine nine one, eight one nought, eight one nine, eight five two.

Examples.

Write down in figures :—

1. Thirty millions, nineteen thousand and four.

2. One hundred and three millions, four hundred and seventy thousand eight hundred and sixty.

3. Three billions and nine.

4. Nine hundred and ninety thousand, nine hundred and nine.

5. Read in words the following numbers :—

80,709,053
100,000,010
90,909,090
9,090,909
170,000
7,777,077

The Roman notation is as follows :—

I. for 1, II. for 2, III. for 3, and IIII. for four. V. is written for 5, X. for 10, L. for 50, C. for 100, D. for 500, M. for 1,000. In general these symbols are combined by addition ; thus CLXVII. stands for 167, MDCLXXVI. for 1676, the largest number being written first ; but there is an exception in the cases of 9 and 90, which are written invariably as IX. and XC., and frequently in the case of 4 and of 40, which are occasionally written as IV. and XL. , thus XIX. stands for 19, XCIX. (not IC.) for 99, and CXLIV. for 144. The notation of higher numbers among the Romans appears to have been very cumbrous and scarcely settled. It was all but useless for the purposes of calculation, and was not used with that object by the Romans themselves.

The following signs are made use of both in arithmetic and algebra :—

Signs of Operation.

+ (plus) shows that the number before which it stands is to be *added* ; thus 3+4 (read as three plus four) means that 4 is to be added to 3, making 7. 5+2 (read as five plus two) means 2 added to 5, also making 7 ; and generally $a+b$ means that b is to be added to a, whatever a and b may mean.

− (minus) shows that the number before which it stands is to be subtracted ; thus 13−5 (read as thirteen minus five) means that five is to be subtracted from 13, making 8. 15−7 (read as fifteen minus seven) means that 7 is to be subtracted from 15, also making 8 ; and generally $a-b$ means that b is to be subtracted from a, whatever a and b may mean ; but the number to be subtracted must always be less than that from which it is to be taken, otherwise we fall into an arithmetical absurdity.

× (into) shows that the numbers between which it stands

are to be multiplied. Thus 3×4 (read as three into four) means that 3 is to be multiplied into 4, making 12. 6×2 (read six into two) means that 6 is to be multiplied into 2, also making 12; and generally $a \times b$ means that a is to be multiplied into b, whatever a and b may mean. Sometimes a full stop at the bottom of the figure is used for this; thus 2×7 or 2.7 are both used to express twice seven.

$\div$ (by) means that the number which stands before it is to be divided by the one which follows it. Thus $6 \div 2$ (read six by two) means that 6 is to be divided by 2, making 3; and $12 \div 3$ (read twelve by three) means that 12 is to be divided by 3, making 4; and generally $a \div b$ means that a is to be divided by b, whatever a and b may mean. Instead of writing $\div$ between the dividend and divisor, it is sometimes the practice to write the divisor under the dividend with a line between them; thus $\frac{289}{17}$ instead of $289 \div 17$.

$=$ (equal) means that the numbers between which it stands are equal to each other; that is, have the same arithmetical value, each taken as a whole. For example, what we have written in the last few paragraphs is briefly expressed:—

$$3 + 4 = 7 \qquad 5 + 2 = 7$$
$$13 - 5 = 8 \qquad 15 - 7 = 8$$
$$3 \times 4 = 12 \qquad 6 \times 2 = 12$$
$$6 \div 2 = 3 \qquad 12 \div 3 = 4$$

This short method of expression not only saves much writing, but also speaks to the eye much more clearly than written words. It is easier to look over, easier to keep right, and easier to correct if wrong. We use it as a convenience, not as a necessity, except in so far as we cannot afford to forego anything that helps us.

Examples of Roman Numerals.

6. Write down in common figures the following numbers expressed in Roman numerals—

XCIV.	CXVI.	MDCCLXVII.
XIX.	LIV.	MCCCXLIX.

7. Express the following numbers in Roman numerals—

45, 18, 499, 49, 1870, 555

CHAPTER II.

ADDITION.

THIS is but a shortened method of counting, and accordingly we must go back to counting to see what are the principles upon which it rests. Let us take for instance—

/ / / / / / / / / / / / / / / / /

in which there are *seventeen* strokes. The first evident principle is that we shall not alter this total number seventeen by counting in groups instead of from end to end, provided that we drop none and count none twice; thus we may take it as 10 and 7 or 7 and 10, which we write as 10+7 or 7+10, and we see that we thus get the same result—17—as by counting from end to end. This has led us to a very important observation; namely, that 7+10 is exactly the same as 10+7, and this is not a fact standing by itself, but one depending on the general principle that the order of addition is immaterial and does not affect the result; for the result depends on the whole quantity and not upon what you take first, second, or last. The student may verify this by writing down on his paper any number of strokes that he pleases, and then adding to them (or writing after them) any other number. He will find the different orders of addition only amount to the same thing as putting separating marks in different places; thus—

11 + 6 = 17

/ / / / / / / / / / / / / / / / / = 17

4 + 13 = 17

I have taken 17 as an example, but it is clear that what I have shown for 17 must necessarily hold for any other total which any man may name; and this remark saves us from the necessity of proving it for every number (for which our life would not be long enough); thus 6+8+3, 6+3+8, 3+6+8, 3+8+6, 8+3+6, 8+6+3, are all the same thing; not because they are found to be so by performing the additions, but for the better reason that they are only different ways of cutting up the same whole.

The following table is called the 'Addition Table.' If any number in the first column be added to any number in the first line, the sum or result of the addition will be found in the square at which the column and line cross.

THE ADDITION TABLE.

	1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9	10
2	3	4	5	6	7	8	9	10	11
3	4	5	6	7	8	9	10	11	12
4	5	6	7	8	9	10	11	12	13
5	6	7	8	9	10	11	12	13	14
6	7	8	9	10	11	12	13	14	15
7	8	9	10	11	12	13	14	15	16
8	9	10	11	12	13	14	15	16	17
9	10	11	12	13	14	15	16	17	18

The usual way of setting an addition sum is to write the numbers one under the other, taking care that the units are under the units, and the tens under the tens, thus :—

6	7	23	21
3	8	54	30
			14
9	15	77	65

The casting here presents no difficulties, but it is different when we have such sums as—

$$\begin{array}{r} 27 \\ 38 \\ \hline \end{array}$$

If we add the 7 and 8 we get 15, and if we add the 2 and 3 we get 5; but it will not do to write 515, because that would mean five hundred and fifteen; and it is clear that we have not got two hundreds, much less five. The fact is that the 1 of the 15, and the 5 arising from the 2 + 3, both represent tens, and therefore, instead of five hundred and fifteen, there are five tens, and one ten, and five units, making altogether six tens and five, or sixty-five. The process in detail is as follows—

$$\begin{array}{r} 27 \\ 38 \\ \hline 65 \end{array}$$

8 and 7 are 15; that is to say, one ten and five. I put down the 5 and I *carry* the one ten to the tens column, without however writing it down; then, when I come to add the tens column, I add in that 1; thus, 1 and 3 are 4 and 2 are 6; and I put down the 6 in the tens column accordingly.

A little reflection will show that this contains in principle the whole method of carrying, because we can carry from the tens to the hundreds, and from the hundreds to the thousands in the same manner as we did from the units to the tens, thus :—

576
879
965

2420

The operation is in detail—5 and 9 are 14 and 6 are 20; that is to say, two tens and no units. I therefore put down nought in the units column, and I then have to remember that I must take account by addition of my two tens in the tens column. The work is now 2 and 6 are 8; 8 and 7 are 15; 15 and 7 are 22—twenty-two tens, that is, or two hundreds and two tens—the two tens I put down in the tens column, and I *carry* the two hundreds. The work of the next column is then—2 and 9 are 11; 11 and 8 are 19; 19 and 5 are 24; that is to say, twenty-four hundreds, which I can write straight down as 24. In any case this means two thousands and four hundreds, and if there had been any column of thousands, I must have put down my 4 hundreds and carried my 2 thousands to be added in with the column of thousands, but, there being no such column in the sum set, that is practically immaterial.

Verification of Addition.—The usual verification is to add both upwards and downwards, and see if the sums agree. This is generally sufficient.

If more is required, or if the student cannot get a long column to cast the same way both up and down, he can cut it up and add each portion separately; then add the sums, as in the following example:—

8794	
753	
1649	
. . . . .	11196
837	
744	
2418	3999
15195	15195

It may be worth while to remark that all verification, or

proof, as it is sometimes called, depends upon a result being obtained in two different ways. It frequently happens that work has to be thus done in different ways, and then it proves itself. This is generally the case with book-keeping. Another instance will be found in Example 13 of this section.

1. Add together 37, 2307, 370, and 307.

2. Add together four hundred and nine, forty-nine, ninety-four, four hundred and ninety, and nine hundred and forty.

(3)	4237089 9999999	(4)	3148475 1000009
(5)	123456789 987654321	(6)	570891 429109
(7)	50110991 40990119	(8)	3137592 827016 5280461 6862408 4719539
(9)	875934 5902 301000 35 1974589 39021 5742 8799999	(10)	783795 783795 783795 783795 783795 783795

11. Add together three millions five hundred and nine thousand and thirty-four, forty millions four hundred and ninety thousand four hundred and three, thirty-seven millions four hundred and thirty, thirty thousand four hundred and

six, seven hundred thousand and eight, eight hundred and ninety-five, twenty-two millions seven hundred and twenty-eight thousand eight hundred and sixty-three.

12. Add together eight hundred and seventy-five thousand and thirty-five, five thousand nine hundred and two, three hundred and one thousand nine hundred and thirty-four, one million nine hundred and seventy-four thousand six hundred and ten, thirty-nine thousand, five thousand seven hundred and forty-two, eight millions seven hundred and ninety-nine thousand nine hundred and ninety-nine.

13. Make up the total line and total column in the following table, and verify the gross total.

	A	B	C	D	E	F	Total
a	307	520	193	1484	3701	1002	
b	8954	1398	7054	302	1135	489	
c	1272	8504	175	5421	719	6403	
d	800	2927	783	1352	9197	8983	
e	4152	4005	8937	858	1861	2724	
f	2992	2171	945	1024	576	567	
g	3456	891	1767	908	9273	3099	
h	755	1425	49	8291	8954	6304	
Total							149059 Gross Total

In such an example as this, which very commonly presents itself in business, the gross total should be made up both from the total line and from the total column. These must of course agree, if the addition is done correctly. I have put down the gross total as a check.

CHAPTER III.

SUBTRACTION.

THIS is the inverse of addition. It enables us to answer the question, By how much does one number exceed another? In arithmetic we obtain this answer from our knowledge of addition. Thus we know that 3 and 4 make 7; hence we infer that 7 exceeds 4 by 3; or that if we subtract 4 from 7 we obtain 3. We cannot subtract a large number from a smaller one without falling into a result which is absurd as far as mere arithmetic is concerned, although, as we shall see by-and-by, it is possible to assign a useful meaning to such a result by the use of ideas which are foreign to pure arithmetic. If, again, we subtract from any given quantity a quantity equal to itself, the difference is nothing. It is customary to call the quantity from which the subtraction is to be made, the minuend; the quantity to be subtracted, the subtrahend; and the result of the subtraction, the difference. Thus, then, we have—

$$\text{minuend} - \text{subtrahend} = \text{difference}.$$

We may also write this as—

$$\text{minuend} = \text{subtrahend} + \text{difference};$$

which shows the connection between subtraction and addition. The student will notice that in the last formula the subtrahend and difference are interchangeable; we may also, therefore, interchange them in the first formula, and we thus obtain—

$$\text{minuend} - \text{difference} = \text{subtrahend}.$$

The student should thoroughly familiarise himself with the equivalents of these three forms, thus—

$$7 - 4 = 3$$
$$7 = 4 + 3$$
$$7 = 3 + 4 \text{ and}$$
$$7 - 3 = 4$$

are only different ways of arranging the same statement of fact.

When we come to numbers containing more than one figure, we have a little difficulty, for which we are obliged to have recourse to what is called *borrowing*, and this process has to be explained. Suppose we have to subtract 7 from 52. We cannot subtract 7 from 2, and we are not supposed to know beforehand our result, which would involve our learning the addition table as far as 100 + 9. What we do, then, is in effect to split the number 52 into 40 + 12; then we know that 7 from 12 gives 5, and there remains also the 40, making 45. Again, if we had to subtract 27 from 52, we should in like manner split up the 52 into 40 + 12; and this taking of one ten from the tens column is called borrowing. We then get, 7 from 12 is 5; and we have still to subtract 20 from the remaining 40, which leaves 20; in all 25. That is the whole meaning of the process of borrowing.

As regards the practice, we borrow from the hundreds column for the tens, and from the thousands column for the hundreds, and so on.

Verification of Subtraction.—The best verification is to add the subtrahend and difference. This ought to give back the minuend, or original quantity from which the subtraction was to be made.

Examples of Subtraction.

(1)	(2)	(3)	(4)
87	195	95	121
35	74	78	87

(5)	8429713 7538894	(6)	10000000 4771213
(7)	987654321 123456789	(8)	2309620835334589 1875248284857 72
(9)	4679298142 3679298143	(10)	5865137267 5766248378
(11)	91650242197 90649251288	(12)	1270106851256158 1196398779220936

The simplest way of getting at the result of a set of mixed additions and subtractions is to add the additive quantities, and then (separately) add together the subtractive quantities, and subtract the sums. Thus, if we have—

$$3795-1532-2019+8759-5104$$

we arrange it as follows :—

	1532	
3795	2019	12554
8759	5104	8655
12554	8655	3899 result

The Arithmetical Complement.—There is a very curious and valuable artifice, discovered by Gunter about 1614, by which we may manage this in one column of addition. We will apply it to the foregoing example. There are three subtractive quantities, each of four figures. The method is to subtract each of these quantities from 10,000, and add these differences and the additive quantities all together. Our result will then be too large by 30,000, which we can cut off at leisure. The work stands thus :—

$$
\begin{aligned}
3795 &= 3795 \\
10000 - 1532 &= 8468^{*} \\
10000 - 2019 &= 7981^{*} \\
8759 &= 8759 \\
10000 - 5104 &= 4896^{*} \\
\hline
&\quad 3)3899
\end{aligned}
$$

If the numbers, or any of them, had exceeded four figures, we must have subtracted from 100,000, or from a million, &c., as the case might require, and then cut off the requisite number of hundreds of thousands or millions.

This method is commonly used in logarithmic calculation, and it is found to save time, especially with the help of the following short rule for taking the complements, as the numbers marked * are called. The rule is: *begin at the left hand, and subtract every figure from* 9 *until the last; subtract that from* 10. Thus for 1532, and for 4780 :—

9	9	9	10		9	9	10	
1	5	3	2		4	7	8	0
8	4	6	8	and	5	2	2	0

It is easily seen that this comes to the same as subtracting from 10,000 in the ordinary way.

In some of the following examples there is a difficulty which may be stated in a simple form by means of an example—

$$2 - 7 + 8 - 1$$

If we begin from the left-hand end, our first operation is to subtract 7 from 2, which cannot be done directly; we have recourse, therefore, to a different arrangement—namely, to take all the additions first, and all the subtractions last. We write it therefore as—

$$2 + 8 - 7 - 1$$

After adding the 2 and the 8, we may either subtract the 7

and the 1 in succession, or we may add the 7 to the 1 and subtract their sum from the sum of the 2 and the 8.

Examples of Mixed Addition and Subtraction, and of the Arithmetical Complement.

(1) 8379 − 544 + 47 − 5891 + 182.

(2) 30103 − 47712 + 60206 − 69897 + 77815 − 84510 + 90309 − 95424 + 100000.

(3) 1 − 2 + 4 − 8 + 16 − 32 + 64 − 128 + 256 − 512 + 1024 − 2048 + 4096 − 8192 + 16384 − 32768 + 65536 − 131072 + 262144 − 524288 + 1048576 − 2097152 + 4194304.

4. Find the complement to 100,000 of each of the subtractive numbers in Example 2, and thus verify the result previously obtained.

5. Find the complement to 100,000 of each of the additive numbers in Example 2, and thus verify the same result in another way.

6. Find the complement to 10,000,000 of each of the subtractive numbers in Example 3, and thus verify the result previously obtained.

7. Find the complement to 10,000,000 of each of the additive numbers in Example 3, and thus verify the same result in another way.

CHAPTER IV.

MULTIPLICATION.

THE kind of addition which occurs most frequently is when the same number has to be added to itself a great many times; as, for instance—

$$3+3+3+3+3+3+3$$

Now there is no difficulty in getting these complicated sums by mere addition, except that they occur so frequently and run into such very large numbers that it becomes necessary to find some shorter process which shall give exactly the same result. This process is called multiplication. There are two or three distinct principles involved in the passage from simple addition to the multiplication of high numbers, but after all it remains true that multiplication is only a shortened method of getting at a particular kind of summation, and therefore that, except saving of time and labour, we can get nothing from multiplication which addition would not equally well give us; we are simply shortening our work.

The following is the ordinary Multiplication Table:—

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	66	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	72	84	96	108	120	132	144

It is what is called a double-entry table, and reads either in column or line; each column and each line of the table

is made up by the continual addition of the figure that stands at the head of it. This the student should verify.

The quantity which is to be multiplied is called the MULTIPLICAND; the quantity by which we multiply is called the MULTIPLIER; and the result of the multiplication is called the PRODUCT. MULTIPLIER × MULTIPLICAND = PRODUCT. MULTIPLICAND × MULTIPLIER = PRODUCT.

It is immaterial to the result which we call *multiplier*, and which *multiplicand*. Thus, 5 × 7 and 7 × 5 express the same thing, namely 35. This is best seen as follows:—

* * * * * * *
* * * * * * *
* * * * * * *
* * * * * * *
* * * * * * *

Here are 35 stars (or asterisks, as printers call them). It is evident that the total number—35—is not altered by the way in which we choose to group them in order to count them. Thus, we may either say there are 7 in each line, and 5 lines; or that there are 5 in each column, and 7 columns. Now one of these ways of counting takes the 35 as 5 × 7, the other as 7 × 5. Hence these are simply two ways of arriving at the same result or product.

The same reasoning applies to any two numbers whatever.

But this is not all. Each star may represent not a unit, but some other number; and in this way it is easy to see the truth of the general principle, that *if any number of quantities have to be multiplied together, the order of multiplication is immaterial.*

If the student wishes to be very ready at mental arithmetic, he should learn the extended multiplication table as far as 20 times 20, and he should make this table for himself.

The process of doing multiplication by one figure (called *short* multiplication) is as follows:—

6209738
7
43468166

7 × 8 is 56; that is to say, five tens and six units. As there are no other units to be taken account of, these stand by themselves, and we accordingly write down 6 in the units place, remembering that we have still five tens, got from the units place, to take in. Setting them aside for a moment, we proceed to multiply the tens. 7 × 3 gives 21 tens; to these we have to *carry* the five tens previously obtained from the units, making in all twenty-six tens, or 2 hundreds and 6 tens. These tens now stand by themselves, and we therefore put them down and proceed to multiply the hundreds, and carry the tens, as follows:—7 × 7 is 49, and the 2 carried makes it 51; that is to say, 51 hundreds. We put down the 1 hundred and carry the 50 hundreds as 5 thousands: a continuance of this process enables us to finish the sum.

We can in the same way multiply by any single figure, or, indeed, by any number for which we have learnt the multiplication table. But, if we have only to multiply by 10, we can do it much more easily by remembering that the effect of multiplying by 10 is simply to change units into tens, tens into hundreds, hundreds into thousands, and so forth. This amounts to simply shifting the number we have to multiply one place to the left from the units place, which we do by writing a nought after it. In the same way, if we have to multiply by 100, or 1000, we simply write two or three noughts after the figures as they stand, and the thing is done. Thus:—

7894	5420	3759
10	100	1000
78940	542000	3759000

Before we can go further we require to use a principle already mentioned, namely, that the order of multiplication is immaterial. For example, if we multiply any quantity by 7 and the product by 10, or if we multiply the same

quantity first by 10 and then that product by 7, we shall get the same result in both cases and this result will be the same as we should have got if we had multiplied at once by 70. As an illustration of this I take a simple case :—

3	3	3
7	10	70
21	30	210
10	7	
210	210	

To a person who only knows arithmetic, I cannot give a proof further than by making him observe that he can verify the principle in every case which falls within his knowledge of the multiplication table, and that the principle involved is not limited by the extent to which he has chosen to learn that.

Thus, multiplying by 3 and then by 4 will be found to be, in every case which can be tried, the same as multiplying first by 4 and then by 3 or by 12 (the product of 3 and 4); the *principle* is evidently general. Now when it happens that a multiplier ends with a 0—as, for instance, 70—we can cut the work very short. We have simply to multiply by 7, and then add a nought to the product; for adding a nought amounts to multiplying by 10 :—

```
 6209738
      70
434681660
```

And, in the same way, if we had had to multiply by 700 or 7000, there would only have been two or three noughts instead of one to be written after the product.

When the multiplier consists of many figures, the process is as follows :—

349172		
75083		
1047516	=	3 × 349172
2793376	=	80 × 349172
1745860	=	5000 × 349172
2444204	=	70000 × 349172
26216881276	=	75083 × 349172

The noughts in the products are not put down, because the figures can be put in their proper places, without risk of confusion, without them. It will be observed that the right-hand figure of each of the products comes directly under the multiplying figure by which it is obtained.

It is sometimes advantageous to split up a multiplier; thus, if we have to multiply by 36, it is easier to multiply in succession by 6 and 6 or by 4 and 9 than to multiply by 36 by long multiplication—that is, taking it as 3 tens and 6. In any case we have two rows of multiplication, but in the last case we get an addition into the bargain. In the case of compound multiplication the advantage is very marked. It is then better to take (for example) 43 as 4 × 10 + 3 than to use long multiplication. For instance, to multiply 1087 and 43 we may proceed in either of the following ways:—

1087	1087	1087
43	4	7
3261	4348 = 4 × 1087	7609 = 7 × 1087
4348	10	6
46741	43480 = 40 × 1087	45654 = 42 × 1087
	3261 = 3 × 1087	1087 = 1 × 1087
	46741 = 43 × 1087	46741 = 43 × 1087

In multiplying by such numbers as 99, 499, 998, &c., it is best to take them as 100 − 1, 500 − 1, 1000 − 2, &c. For

example, 99 × 1087 is best found as follows—by subtraction :—

```
108700
  1087
------
107613
```

In former times, when arithmetic was not so familiar, multiplying by 9 was also done by adding a nought and subtracting. This was called *regula pigri*, or the lazy-bones rule.

Examples of Multiplication.

(Remember that the order of multiplication is immaterial.)

(1) 17 × 19 = 323.
(2) 17 × 98 = 1666.
(3) 17 × 908 = 15436.
(4) 107 × 98 = 10486.
(5) 1070 × 9080 = 9715600.
(6) 355 × 755 = 268025.
(7) 34015 × 38175 = 1298522625.
(8) 31622777 × 63245553 = 2000000018760681.
(9) 7 × 11 × 13 = 1001.
(10) 1 × 2 × 3 × 4 × 5 × 6 × 7 × 8 × 9 × 10 = 3628800.
(11) 2 × 3 × 5 × 7 × 11 × 13 × 17 × 19 = 30 × 1001 × 323 = 9699690.
(12) 30102 × 47712.
(13) 4771213 × 602059999.
(14) 2793 × 812358 × 857.
(15) 744615 × 427282 × 15905.
(16) 708421 × 930937 × 461762 × 972744.
(17) 1010101 × 999999 × 1111111 × 9090909.
(18) 2337 × 5501 × 3616 × 9827 × 3543.
(19) 9998 × 9999 × 10000 × 10001 × 10002.
(20) 9999 × 9999 × 9999 × 9999 × 9999 × 9999.

(21) 9999 × 99980001 × 999700029999.
(22) 1234 × 1234 × 1234 × 1234.
(23) 1234 × 2345 × 3456 × 4567.
(24) 123456789 × 987654321.

CHAPTER V.

DIVISION.

THE object of DIVISION is to find how many times one number is contained in another. This *number of times* is called the QUOTIENT.

The first idea of obtaining the result is to use subtraction, and count the times we have to use it. Thus, to find how many times 3 is contained in 14:—

```
14
 3 . . (1)
11
 3 . . (2)
 8
 3 . . (3)
 5
 3 . . (4)
 2
```

We see we can take 3 away from 14 *four* times in succession, and then we leave 2. But if we had helped ourselves by the multiplication table (of *three* times), we might have done it more shortly. For since $5 \times 3 = 15$, 3 will go 5 times into 15, exactly. Therefore 3 will not go 5 times into 14. Again, $4 \times 3 = 12$, and thus 3 will go 4 times into 14, and leave something over. This 'something over' is evidently $14 - 12$, or 2.

The quantity to be divided is called the DIVIDEND, the quantity by which we divide is the DIVISOR, the *number of times* is the QUOTIENT, and what remains over (if any such there be) is called the REMAINDER.

DIVIDEND = DIVISOR × QUOTIENT + REMAINDER.

Suppose we have to divide 74 by 3. We do not find 74 in our multiplication table of *three times*. What then are we to do?

Observe that 74 is 7 tens and 4. Beginning with the tens, 3 goes into 7 twice and 1 over; that is to say, dividing 7 tens by 3 gives 2 tens for quotient, and 1 ten for remainder. Hence 74 divided by 3 gives 2 tens, and remainder 14 units. But this is too large a remainder, for 3 will divide it, giving quotient 4, and remainder 2. The total quotient is thus 2 tens and 4, and the final remainder 2, and so we find—

$$74 = 3 \times 24 + 2.$$

In practice we arrange the sum, and work it, as follows:—

3) 74 (2
   24

Three into seven goes twice and one over. We put down 2 and carry 1. Three into fourteen goes four times and two over. Put down 4. We do not carry the 2 because it is two units, and there is nothing to carry it to, but we put it down as the remainder.

In the same way we have—

Divisor 3) 1458074 (2 remainder—
             486024 quotient

When the divisor is a small one, as in this case, we put down nothing but the quotients as we go, doing the subtraction in our heads. But when the divisor is large (as 5783, for instance), there are two difficulties. In the first place, we do not know the multiplication table up to 5783; and secondly, even if we did, we should hardly be able to do the subtractions without putting them down. There are

two ways out of the difficulty. The first is to form a multiplication table up to 9 times 5783. This is best done by continued addition, which ought to verify itself at 10 times:—

1 . .	5783
2 . .	11566
3 . .	17349
4 . .	23132
5 . .	28915
6 . .	34698
7 . .	40481
8 .	46264
9 . .	52047
10 . .	57830

This is the best plan when there are very many figures in the quotient.

Suppose the dividend to be 108419716121, we arrange the sum as follows:—

5783) 108419716121 (18748005 quotient

Quotient figures:		
(1) .	5783	
	50589 . . .	1st remainder with [next figure
(8) .	46264	
	43257 . . .	2nd "
(7) .	40481	
	27761 . .	3rd "
(4) .	. 23132	
	46296 . .	4th "
(8) .	. 46264	
	321 . .	5th "
☞ (0) . . .		
	3212 .	6th "
☞ (0) . . .		
	32121 .	7th "
(5) . . .	28915	
	3206	final remainder.

I have inserted the steps marked ☞ merely to show the principle. In practice we simply put down the two noughts in the quotient, and go at once to 32121 for the division.

It is more usual, however, to form the products as we want them, instead of going to the trouble of making out the table. We then have to make a guess at the quotient figure at each step, by using the left-hand figure of the divisor alone to give us a trial quotient figure. Thus at the third step our dividend is 43257, and 43 divided by 5 gives 8 as a trial quotient. But, on trial, we find it is too much—namely, 46264. We therefore try 7 times, and find that that will do. A little work is sometimes lost in this way, but it is generally shorter, on the whole, than making out a special table.

On the continent of Europe it is usual to shorten the writing by setting down the remainders only and not the products; the multiplication and subtraction being worked together: thus—

```
5783 ) 1084197 ( 18
        50589
         4325
```

The first step is simple subtraction, giving 5058 for remainder. The work of the next step is as follows:—

8 times 3 is 24: from 9 gives 5 (which put down) and carry 2.
8 times 8 and 2 gives 66: from 8 gives 2 (which put down) and carry 6.
8 times 7 and 6 gives 62: from 5 gives 3 (put down) and carry 6.
8 times 5 and 6 gives 46: from 50 gives 4 (put down).

It sometimes happens that *one* has also to be carried from the subtraction. For instance, in this case—

```
5783) 50581 (8
       4317
```

we say:—8 times 3 is 24: from 1 gives 7 (put down) and carry 3 (instead of 2). Then 8 times 8 and 3 gives 67: from 8 gives 1 (put down) and carry 6, &c., &c.

This is the proper way to work division for ordinary purposes. The common English fashion of putting down all the quotients is clumsy work, and takes up needless time and paper.

Our division sum, worked in the best way possible, would stand as follows :—

```
Dividend  108419716121 ( 5783 divisor
           50589          18748005 quotient
            43257
             27761
              46296
                32121
                 3206 final remainder
```

It is sometimes possible to substitute two or three short division sums for one long division. For instance, if the divisor be 252, we may divide in succession by 3, 7, and 12, or by 4, 7, and 9, or by 6, 6, and 7. For it is evident, since

$$4 \times 7 \times 9 = 3 \times 7 \times 12 = 6 \times 6 \times 7 = 252$$

that if we multiply any given quotient by either of these sets of multipliers, we shall get the same product; and therefore, that if we divide any given product by these, we must get the same quotient. In most cases this consideration saves labour, and especially so in compound division.

But if the divisor will not split up into exact divisors, we cannot proceed as in multiplication. To multiply by 43 we can take it as $6 \times 7 + 1$, or as $4 \times 10 + 3$; but this does not help us in division.

In dividing by 10, 100, 1000, &c., it is sufficient to mark off one, two, or three, &c., figures from the right-hand end.

So also to divide 108479 by 300 we work as follows :—

```
3) 108479 | 23 (223 remainder
   36159 quotient
```

So long as we pay no attention to remainders, it is immaterial in what order we perform a series of multiplications and divisions. Thus, if we have to multiply successively by 7 and 5 and to divide by 3 and 11, we may perform these processes in any order we find convenient, so far as regards the quotient. But it requires some knowledge of compound arithmetic, or fractions, to know what to do about the remainders in these cases.

The proof of division is by multiplication, adding in the remainder, if any.

Examples of Division.

(1) 3) 17183219 (

(2) 7) 995742 (

(3) 8) 20176325 (

(4) 11) 1000000 (

(5) Divide 87911123 by 7070.
Answer 12434 ; remainder 2743

(6) 77) 87911123. Prove by dividing by 7 and by 11 in succession.
Answer 1141702 ; remainder 69.

(7) 10000000000000000 ÷ 967
$=10341261633919\frac{187}{967}$

(8) 87963577784034 ÷ 99999
$=879644457\frac{48491}{99999}$

(9) 575405478000888 ÷ 718367
$=800990967\frac{9999}{718367}$

10. Find from this the quotient of
5754054870008880 ÷ 718367000
(neglecting the remainder).

11. Divide 1000000000 by any of the following numbers :—

9	99	999	9999
11	. 101	1001	10001
13	37	61	377
7	91	134	512
25	625		1953125

12. Divide 31415926536 by 648.

13. Divide 31415926536 into 100 . . . . . with as many noughts added as may be necessary to give ten figures in the quotient.

14. Do the same with 2302585093.

15. Do the same with 4342944819.

CHAPTER VI.

THE PROOF OF NINES.

A VERY useful proof of multiplication and division and any combinations of these operations, with or without addition or subtraction, is obtained by what is called casting out the nines. It is indeed true that the verification which this gives is not absolute, inasmuch as it will not detect an error which can be divided by nine, or compensating errors, or a mere transposition of figures. Nevertheless it is a great help in practice.

The remainder of any number given by division by 9 is the same as the remainder that we should get by adding together the figures of which the number consists and dividing that by 9. For instance, take 37248751. The sum of the figures is 37, and the sum of these two figures is 10, and of these again 1. Now if we divide any of these numbers (namely, 37248751, 37, or 10) by 9, it will be found that the remainder is 1 for all of them. Hence we may obtain the remainder of any number to the divisor 9 by

continually adding up the figures of which it consists, and we may, if we please, cast out any nines as we go.

The application of the principle is this. Take any series of numbers and find their remainders to 9. Combine the numbers in any way you please, for one result; and combine the remainders in the same way to form another result. These two results will give the same remainder to 9.

Thus, for addition :—

8277	remainder to 9	6
3485	"	2
7146	"	0
8036	"	8
26944	"	16

and these results each give 7 as the remainder to 9.

Again for subtraction :—

87235	remainder to 9	7
14565	"	3
72670	"	4

Again for multiplication :—

$$349751 \times 28637 = 10015819387$$

The remainders of the two first numbers are 2 and 8. The product of these is 16, and the remainder of this to 9 is 7. The remainder from the product 10015819387 is also 7, and this coincidence is regarded as proving the multiplication. As already remarked, the proof is not absolute, because there may be an error of 9, or of any whole number of nines; but this is a class of error which is not commonly met with in practice.

The method may even be used in detail to verify each step of the sum, as follows :—

349751	remainder	2
28637	"	8
2448257	"	5
1049253	"	6
2098506	"	3
2798008	"	7
699502	"	4
10015819387	"	7

thus checking every separate multiplication and the addition as well as the complete result.

In the case of division the process is less direct. For instance, if we divide 417 by 29 the quotient is 14 with remainder 11. The most convenient form in which to apply the proof of nines is to write this in the form $29 \times 14 + 11 = 417$. The remainders give $2 \times 5 + 2$ or 12. The remainder to 9 both of this and of the dividend 417 is 3, which therefore proves the work.

The proof is chiefly useful for multiplication or division, or for the combinations of these with addition or subtraction. Mere subtraction and addition are so easily verified with certainty and accuracy that it is hardly worth while to apply this verification to them.

The principle depends upon the fact that a number consisting of a single figure followed by any number of ciphers gives that figure as the remainder to the division of that number by 9. Thus 700 divided by 9 has 7 for its remainder.

The rule does not apply to any but decimal arithmetic. Thus in *l. s. d.* the 12*d.* and the 20*s.* throw out the rule of nines.

In very heavy multiplications or divisions it is sometimes worth while to try every separate step by the rule of nines, before going to the next step. In this way the work is always kept right, instead of mistakes being left until the result comes to be checked.

For examples in the proof of nines, check the answers given to the questions in the rules of multiplication and division.[1]

CHAPTER VII.

THE COMPOUND RULES

Section I.—Reduction.

Hitherto we have been dealing with mere *numbers*; we have now to apply these to *things*. We have two principal cases to consider.

1. Where each of these *things* is an individual which can be neither divided nor combined with others, as is the case with men, animals, ships, houses, and the like. In this case we have an absolute unit, from which we *must* reckon.

2. Such cases as the reckoning of time, value, size, weight and so forth, in which the unit is really arbitrary, and for each of which practically we use several units. The act of passing from one unit to another is what we call Reduction. Thus in money, 2*l.* 15*s.* is 22 half-crowns, or 55 shillings, or 660 pence, or 2640 farthings, as well as two pounds and three-quarters. The object of reduction is to be able to express a given amount of money, named in any one of these ways, in any other. For this purpose it is first necessary to learn the tables of money, weights, measures, &c., or at least to understand them and to have them ready for reference. They will be found at the end of this volume.

The simplest illustration of the principle of reduction is

[1] The simplicity of the rule of nines depends upon the number 10 (=9+1) being chosen for the scale of notation. In any other scale of notation, there is a similar rule for the number, one less than the scale.

There are also more complicated rules of a somewhat similar character for ascertaining the remainder of a number divided by 7, 11, or 13. These depend upon 11 being 10+1, and upon 7 × 11 × 13 being 1000+1. They are mere curiosities, and quite useless in practice.

such a case as occurs when we pass from shillings to pence or from pence to shillings.

Thus there being twelve pence in a shilling, it is evident that any number of shillings contains twelve times that number of pence, and hence that to pass from shillings to pence we must multiply by 12; and consequently, to turn pence into shillings we must divide by 12. Thus seventeen shillings is 17 × 12, or 204 pence; and again, 175 pence is 175 ÷ 12, or fourteen shillings and seven pence. Here the remainder 7 indicates that there is not a round number of shillings in 175 pence; but that there are seven pence over and above the fourteen shillings. These remarks really contain the whole principle of reduction. All the rest is mere convenience of arrangement and tricks for shortening the work.

Reduction in the Decimal System.

This involves no actual arithmetical work, but merely a shift of the unit point. Thus 15 dollars 35 cents are simply 1535 cents, and cents are reduced to dollars and cents by simply cutting off the last two figures for the odd cents. Thus 2683 cents are 26 dollars 83 cents. So again in measures of length: 84 kilomètres 374 mètres are 84,374 mètres; and again, 7657 mètres are $7\frac{657}{1000}$ kilomètres. This is the chief advantage of the decimal system of measures—that the reductions are done at sight.

English Money.

```
28l. 17s. 6½d.
 20                4) 27722 ( 2
 ---                  -----
577               12)  6930 ( 6
 12                   ------
 ---              20)   57,7
6930                   -----
   4                    28,17
-----
27722
```

In this example we have a mixed sum embracing four units

of account, and we propose to express the whole amount in farthings.

We do this step by step. We first reduce the pounds to shillings, multiplying the number of pounds by 20. For shortness, we add in the odd 17 shillings as we go; that gives us 577 shillings; we then get the pence by multiplying this by 12, and adding in the odd 6 pence. In like manner, to get the farthings, we multiply the 6930 pence by 4, and add in the halfpenny as 2 farthings, making 27,722 farthings.

To convert the farthings back into pounds, we have simply to reverse the process, and to divide in succession by 4, 12, and 20, the remainders giving the odd money of the lower units of account, or denominations, as they are sometimes called.

We might, if we liked, have passed at once from farthings to pounds, or pounds to farthings, by dividing or multiplying by the number of farthings in a pound—viz., 960. Thus—

```
960) 27722 (28
     1920
     ----
      8522
      7680
      ----
       842
```

We should then have the remainder 842, in which there would be odd shillings and pence. We may get the shillings by dividing by 48, the number of farthings in one shilling. Thus—

```
48) 842 (17
    48
    ---
     362
     336
     ---
      26
```

We have now a remainder from the shillings of 26 farthings, or $6\frac{1}{2}d.$, and we have obtained the same value as before—namely, $28l.\ 17s.\ 6\frac{1}{2}d.$ This process is quite correct in principle, but it is not generally so convenient in practice as the plan of taking the denominations successively.

Weights and Measures.

The reduction of weights and measures is done on exactly the same principle, only that the multipliers or divisors by which we pass from one to the other are no longer 20, 12,

and 4, but those belonging to the particular table. Thus for avoirdupois weight, the reduction of 2 tons 14 cwt. 3 qrs. 24 lbs. to ounces, and the inverse sum of reducing 98,496 oz. to tons, &c. are as follows :—

2 tons 14 cwt. 3 qrs. 24 lbs.	
20	
54 cwt.	4) 98496 oz.
4	4) 24624
219 qrs.	7) 6156 lbs.
7	4) 879 | 3 rem^{r}.
1533	4) 219 | 3 rem^{r}.
4	20) 5.4 | 3 qrs. rem^{r}.
6156 lbs.	tons 2,14 cwt.
4	
24624	
4	
98496 oz.	

In the reduction from lbs. to qrs. there is a little difficulty about the remainders. Instead of dividing by 28 at once, we divided by 7 and by 4. Now if we had any name for the 7 lbs.' weight, we should have been in no difficulty; we should have had 3 *lbs.* and three 7 *lb.* weights. We get rid of the difficulty by writing this down all together as 24 lbs. This apparent difficulty frequently arises in reduction and in the division of mixed quantities. But a little attention to the principle mentioned above is always sufficient to clear it up.

Reduction of time.—Much confusion frequently arises from want of precision in stating the different units of time, and the first point to look to in these questions is to see what is meant, or if you cannot find that out, to state clearly what you have *taken* in working for your answer. For example,

a month may mean anything from 28 to 31 days, and therefore if I ask you how many months there are in 3652 days (for example), you must first ask me what months I mean. There are exactly 120 calendar months in that period, counting two leap-years. But if we reckon a month as four weeks, there are 130 months and 12 days, and no proper answer can be given to the question without first settling what is to be understood by a month.

If there be no statement to the contrary, it is usual to reckon 365 days to a year and 12 months to a year. If months and days are mentioned, a month is taken at 30 days. If months and weeks are mentioned, a month is taken at 4 weeks. Of course this is inexact on the face of it.

Square Measure: Land Measure.—It is important to bear in mind that a square yard is a piece of ground three feet wide and three feet long, and therefore that to pass from square yards to square feet, we must multiply by 3 twice or once by 9. For if we only multiply by 3 once, our reduced unit will not be a square foot, but a strip three feet by one foot. Similarly to reduce poles to square yards we must multiply twice by 5½, which is the number of *yards long* in a pole, or once by 30¼, which is the number of square yards in a square pole.

Cubic Measure.—In this case we have to make a third multiplication or division—for thickness—as well as the two for length and breadth. Thus 5 cubic feet are 5 × 12 × 12 × 12 cubic inches, or 5 × 1728, or 8640 cubic inches, while 5 linear feet are only 60 inches.

The *proof* of reduction is to invert the process so as to work back to what we started from.

Fractional reductions.—Reduction cannot be completely understood without some knowledge of fractions. For instance, in reducing poles to square yards, we have to multiply by 30¼. This difficulty may be got over in general by reducing to some lower denomination and back again. Thus

the number of square inches in a pole is 39,204, and the number of inches in a square yard is 1296. We can thus reduce the poles to inches, and then reduce the inches to yards without using fractions. The better way, however, is as shown in the following example :—

$$\begin{array}{r} 176 \text{ poles} \\ 30 \\ \hline 5280 \\ 44 \\ \hline 5324 \text{ square yards} \end{array}$$

Examples of Reduction.

1. Reduce 137*l.* 9*s.* 11*d.* to halfpence.
2. How many halfcrowns are there in 21,711 farthings?
3. Reduce 145 tons 14 cwt. 3 qrs. 8 lbs. to ounces.
4. How many tons, &c. are there in a million of ounces?
5. How many weeks have elapsed from the birth of Christ to the present date?
6. What is the number of seconds in a year?
7. Reduce 10,000,000 seconds to years.
8. Reduce 157 lbs. 7 oz. 8 dwts. to grains.
9. Reduce 78 lbs. 10 oz. 5 dr. (apoth. wt.) to grains.
10. Reduce 1,000,000 grains to lbs. troy.
11. How many yards are there in 18 miles 3 furlongs 1 pole?
12. Reduce 100,000 feet to miles, &c.
13. How many seconds are there in a semicircle?
14. What is the number of square feet in a square mile?
15. What is the number of cubic inches in a cubic yard?
16. How many cubic feet are there in a million cubic inches?
17. How many guineas are there in 444,912 farthings?
18. How many half-crowns are there in 1000 guineas?

19. How many half-pints are there in a kilderkin of ale?

20. Reduce 143 a. 2 r. 37 p. (land measure) to square feet.

21. Reduce 14,695 square yards to a. r. p.

22. How many pounds avoirdupois are there in 21,000 pounds troy [reduce to grains and back]?

23. Supposing a cubic foot of water to weigh 1000 ounces avoirdupois, how many troy grains are there in a cubic inch?

24. How many half-pints are there in a butt of sherry?

25. A gallon of water measures 277 cubic inches very nearly, and weighs 10 lbs. avoirdupois. How many cubic feet and inches are there in a ton of water?

Section II.—ADDITION.

Where the scale of denomination is decimal, the work is exactly the same as for the simple rules. We have simply to take care not to lose sight of the unit point; that is to say (for instance), not to confuse dollars with cents, or with tens of cents. In this case, in fact, we pass from the column of tens of cents to that of dollars, in identically the same way as we pass from the tens to the hundreds column in simple arithmetic.

In the following examples it will be seen that the addition proceeds exactly as in simple arithmetic:—

Francs centimes.	*Dollars cents*
75·54	12·65
125·45	1·15
·50	155·75
1·75	·5
350·	271·
2·25	12·25
·14	120·
555·63	572·85

When we are dealing with pence and shillings we can no

longer do this, because we pass from pence to shillings with the help of the divisor 12 instead of 10. If then we have got, say 90 pence, as the result of our operation on the pence column, we are no longer at liberty to carry 9 to the shillings column; but since 90 pence is 7*s.* 6*d.*, we must put down the 6 pence and carry 7 to the shillings. There is no other difference in principle between the simple and compound rules. In practice, however, the mixture of decimal arithmetic with divisions which are not decimal is puzzling to beginners, and a source of inconvenience and mistake to all.

Rule for Addition.

Arrange the quantities to be added in lines one underneath the other, taking care that each column is not only right in respect of tens and units, but is also of the same denomination That is to say, we must not get 3 pence under 2 shillings, and add them together as 5. This settled, begin with the lowest denomination and add. Reduce the sum by means of the proper divisor, putting down the remainder, and carrying the quotient to the next higher denomination, and so on. Thus—

£	*s*	*d.*
257	18	7
124	11	10
89	17	6
11	0	5
87	7	0
£570	15	4

On adding the pence column we find it comes to 28 pence. This is 2 shillings and 4 pence. We therefore put down the 4 pence and carry the 2 shillings. With this carrying figure we find the shillings amount to 55, or 2*l.* 15*s.* We put down the 15 shillings and carry the 2*l.* The rest is simple addition.

Tons	*cwts.*	*qrs.*	*lbs.*
25	14	0	21
37	0	3	11
	4	2	7
19	8	2	7
145	11	3	0
227	19	3	18

Adding up the lbs. we get 46 lbs., which is 1 qr. and 18 lbs. We put down the 18 lbs. and carry one quarter. With this carrying figure the quarters amount to 11, or 3 qrs. to put down and 2 cwt. to carry The cwt. column gives 39, or 1 ton 19 cwt Putting down the 19 cwt we have now to carry 1 ton, and the rest is simple addition

The proof or verification is the same as for simple addition.

Compound Addition.

Examples.

(1)

£	*s.*	*d.*
237	18	4½
19	7	11¼
154	8	7
22	10	0¾
37	17	8
24	5	9¾
172	13	2½
109	19	8¼

(2)

tons	*cwts.*	*qrs.*	*lbs.*	*oz*
141	7	2	17	4
72	15	3	12	13
	14	0	8	0
189	19	1	27	8
275	4	2	5	15
190	0	0	17	14

(3)

lbs.	*oz.*	*dwts.*	*gr.*
14	3	14	21
8	7	11	9
2	11	5	18
3	0	18	4
21	6	0	0
19	8	0	7

(4)

acres	*r.*	*p.*
157	3	37
169	2	5
214	2	28
147	1	18
94	3	1
182	0	11

(5)

miles	yards
17	817.
9	348
11	1520
28	714

(6)

yards	qrs.	in.
159	2	7
245	3	4
95	1	8
77	2	5
143	0	2
25	3	7

(7)

feet	in.	10ths.
1	9	8
	11	3
2	4	5
3	3	9
3	8	0
3	0	5

(8)

sq. ft.	sq. in
17	81
8	132
179	50
25	5
118	111
37	54

9. Add together 127 dollars 25 cents, 13 dollars 75 cents, 45 cents, 150 dollars 65 cents, 1030 dollars 10 cents, and 14 dollars 95 cents.

10. Add together 115 francs 65 centimes, 39 fr. 55 cent., 85 cent., 142 fr. 45 cent., 1572 fr. 85 cent.

11. Add together 1000 tons 1000 cwt. 1000 qrs. 1000 lbs. and 1000 oz.

12. Add together 10 miles 100 furlongs 1000 poles 10,000 yards 100,000 feet and 1,000,000 inches. Give the answer both in miles and in feet and inches.

13. Add together 10 acres, 90,000 yards, 11,937 feet and a million inches. State the answer in acres, chains, yards, &c.

14. Add together 13 cubic yards, 157 cubic feet, and 10,000 cubic inches. State the answer in yards, feet, and inches, and also in inches.

15. Fill up the last column and the last line in the following table: —

	A			B			C			D			E			Total		
	£	s.	d.	£	s.	d	£	s.	d.	£	s.	d.	£	s	d.	£	s.	d.
a	31848	7	0	260268	11	7	267282	6	1	270271	13	9	74740	7	6			
b	4028	4	4	44137	14	0	89904	6	9	71528	6	10	16906	1	10			
c	495	11	8	14833	10	6	11781	18	4	19840	14	4	3544	11	3			
d	18426	17	10	28417	7	6	48373	4	7	...			4419	6	9			
e	423	19	0	13264	19	9	25559	15	5	..			3424	7	11			
f	116	6	6	2655	18	7	2165	17	9	...			764	2	3			
g	25	8	6	2604	1	9	1609	15	11	..			163	4	7			
Total																		

The gross total is 1,333,887*l*. 0*s*. 4*d*.

Section III.—Subtraction.

The remarks made about addition apply to subtraction, with the exception that we have there to borrow instead of to carry. Evidently when we borrow from a higher denomination, that will not necessarily give ten; in fact, 1 borrowed from shillings to enable us to subtract the pence column will be 12 pence, and not 10 pence; and so 1*l.* borrowed from the pounds column, to enable us to subtract the shillings, will be 20, and not 10.

£	*s.*	*d.*
350	4	5
117	7	9
232	16	8

tons	*cwts.*	*qrs.*	*lbs.*
21	7	1	7
9	15	3	21
11	11	1	14

Proof.—The proof or verification is to add the difference to the subtrahend. This will give the minuend if the work is right.

If there are mixed additions and subtractions, all the additive quantities should be added separately, and all the subtractive quantities should be added together. One subtraction will then give the answer.

Examples.

1. My wages for the month are 10*l.* 6*s.* 8*d.*, but I have had 7*l.* 11*s.* 10*d.* advanced to me. How much have I still to draw?

2. A turnpike gate is 3 miles 1659 yards from the town, and an inn is 4 miles 706 yards from the town by the same road. How far is it from the turnpike to the inn?

3. John Jones will be 41 years, 37 weeks, and 1 day old on June 17, 1873. Neglecting leap years, what day was he born?

4. Give the answer to the question, counting leap years, which come every fourth year, counting back from 1872.

5. The gross weight of a package is 7 cwt. 1 qr. 5 lbs., and the tare[1] is 3 qrs. 17 lbs. What is the net weight of the articles contained in it?

6. A full bottle weighed 11 oz. 4 dwt. 11 grs. Its empty weight was 4 oz. 18 dwts. 15 grs. What was the weight of the liquid which it contained?

7. On the 1st of August I earned 1*l.* 11*s.* 6*d.* and spent 2*l.* 13*s.* 4*d.*; on the 2nd I earned nothing and spent 11*s.* 6*d.*; on the 3rd I earned 4*l.* 13*s.* and spent 1*l.* 8*s.* 6*d.*; on the 4th I earned 5*l.* 17*s.* and spent 19*s.* 10*d.* By how much, on the whole, am I richer on the evening of the 4th than on the morning of the 1st?

8. (See last question.) What is the least capital I could have had to start with, if I never was in debt?

9. A man walks due east 25 miles 1130 yards, then turns back and walks west 10,000 yards; then east 5 miles 120 yards; then west 17 miles 970 yards, and then east 6000 yards. How far will he be from the point from which he started?

[This question may be done either by ascertaining the position of each halting place in succession, or more shortly by adding all the eastings together, and all the westings together, and subtracting.]

Section IV.—Multiplication.

By this we understand the multiplication of a mixed quantity, such as *l. s. d.*, or tons, cwts. qrs., by a mere numerical multiplier. To multiply money by money has no

[1] Tare is the weight of the packing-case, straw, and other belongings.

meaning. It is true that we may multiply length by length, and this will give us square measure; and that we may multiply length by weight, which gives us *leverage* or *moment*, but these are not the objects of the present rule. They will be considered further on.

Compound multiplication presents no difficulties We proceed as in simple multiplication, only taking care that in carrying from one denomination to another we use the proper divisor instead of simply carrying the tens figure. Thus —

£	s	d
75	18	7
		7
531	10	1

For the pence, 7 times 7 is 49, or 4 shillings and a penny We carry 4 shillings, and put down the penny 7 times 18 are 126, and 4 are 130, or 6*l* 10*s* We put down the 10 shillings, and carry the 6*l* The rest is simple multiplication

When the multiplier is considerable, but not very large, we may split it up and do the multiplication piecemeal. For instance, $47 = 4 \times 10 + 7$, and if we have to multiply 112 acres 3 roods 37 poles by this number, the work is as follows —

Acres	*roods*	*poles*	
112	3	37	
		4	
451	3	28	. . 4 times
		10	
4519	1	0	. . 40 times
790	3	19	. . 7 times
5310	0	19	. . 47 times

When, however, the mutiplier is very large this is no longer convenient, and there are then three ways of doing the thing.

1. By *practice*, as it is called. This is a separate rule requiring a knowledge of fractions. It will be explained further on.

2. By reducing the multiplicand to its lowest denomination; then performing the simple multiplication, and afterwards reducing back. Thus in the foregoing example: 112 acres 3 roods 37 poles is 18,077 poles. Multiplying this by 47, we get 849.619 poles, or 5310 acres 0 roods 19 poles.

3. Another method is to multiply each denomination separately, and reduce as we go. For this purpose it is usually most convenient to make the different parts of the compound number the multipliers. For instance, if we have 24*l.* 18*s.* 7*d.* to be multiplied by 4317, we may proceed as follows:

4317	4317	4317
24	18	7
17268	34536	12) 28219
8634	4317	20) 235,1 7
103608	20) 7770,6	117,11 7
3885 6	3885	
117 11 7		
£107610 17 7	answer	

A multiplication of this sort is, however, done much more shortly by the rule called Practice.

The proof of multiplication is simply to do it over again in some different way.

Examples.

1. 17*l.* 18*s.* 11*d.* × 7 = 125*l.* 12*s.* 5*d.*
2. 251*l.* 13*s.* 7*d.* × 11 = 2768*l.* 9*s.* 5*d.*
3. A frigate is armed with 14 sixty-eight pounders weighing 95 cwts. 3 qrs. 7 lbs. each, and 26 thirty-two pounders weighing 64 cwts. 1 qr. 14 lbs. each. What is the weight of her armament?

4. What will a rate of 1*s.* 11*d.* in the pound come to on a rental of 2845*l.*?

5. Each sailor being supposed to weigh 1 cwt. 1 qr. 9 lbs., what is the weight of a crew of 861 men?

6. 5 tons 17 cwts. 3 qrs. 17 lbs. 5 oz. × 43.

7. 4 lbs. 8 oz. 17 dwt. 18 grs. × 24.

8. 19 miles 3 furlongs 105 yards × 17.

9. 579 acres 3 roods 23 poles × 19.

10. 17 square feet 9 inches × 179.

11. 18 yards 2 qrs. 3 nails × 14.

12. 1 cubic yard 15 feet 1124 inches × 7.

13. Telegraph poles being placed 132 feet apart, what distance will 1235 poles do for?

14. The circumference of a wheel is 16 feet 1½ inches. How many miles will it run in 125,375 revolutions?

15. The pitch of an endless screw is half an inch, and it is driven at the rate of 150 turns a minute. What is the edge velocity of the wheel in feet per hour?

16. In a multiplying train of wheels, a driver of 3 inches radius acts on a ½-inch pinion keyed to a 3-inch wheel. The second wheel will therefore move 6 times as fast as the first. Supposing there to be 5 such wheels and pinions following a driver which makes 40 revolutions a minute, how many revolutions will the last make in an hour?

17. An express train moves eight times as fast as a man who walks 7 feet in a second. What is the speed of the train in miles per hour?

18. A box of figs weighs 3 lbs. 5 oz. What is the weight of 500 dozen boxes?

19. What is the weight and cost of a square mile of glass weighing 9 oz. and costing fourpence-halfpenny a square foot?

20. What is the weight of a million of bricks at 4 lbs. 2 oz. each, and the cost at 27*s.* 8*d.* per 1000?

Section V.—Division.

Proceed as in ordinary division, beginning with the highest denomination, until you come to a remainder less than the divisor. Then reduce that remainder to the next lower denomination, adding in the odd figures, if there be any, and then continue the division until you again come to a remainder which requires reduction, and so on. Thus to divide—

```
      Tons   cwts.  qrs.  lbs.
   7) 1080    12     3    21
      ----   ---    --    --
       154     7     2     7
```

Dividing the tons by 7, we have 154, with remainder 2. This remainder is 2 tons, or 40 cwt., which, with the odd 12 cwt., makes 52 cwt. Dividing this we get 7 cwt., and a remainder of 3 cwt., or 12 qrs. We have now 15 qrs. to divide This gives 2 qrs. and remainder 1 qr. or 28 lbs. With the 21 lbs. this gives 49 lbs. The division of this gives 7 lbs.

The process is not different whatever the divisor may be; but a large divisor will require long division instead of short. Thus to divide 23,343*l.* 17*s.* 9*d.* by 365, we proceed as follows :—

```
        £        s.    d.
  365) 23343     17    9 ( 63
       2190
       ----
        1443
        1095
        ----
         348
          20
        ----
        6977 ( 19
         365
        ----
        3327
        3285
        ----
          42
          12
        ----
         513 ( 1
         365
        ----
         148
           4
        ----
         592 ( 1
         365
        ----
         227           = 63l 19s. 1¼d.
                remainder 227 farthings.
```

The only points calling for notice in this division are where the multiplications come in. The reason for them is easily seen.

Omitting the products, and doing the multiplication and subtraction at once, which, as has been already stated, is the better way, the process is as follows :—

```
23343—17—9 ( 63
1443
 348
  20
6977 ( 19
5327
  42
  12
 513 ( 1
 148
   4
 592 ( 1
 227
```

Where the divisor can be split up into component parts or *factors*, it is sometimes preferable to substitute two or three short divisions for one long division. Thus, to divide by 56, use 7 and 8 in succession.

The only proof of compound division is either to do it in another way, or to get back the dividend by multiplication, adding in the remainder, if any.

Examples.

1. A man leaves 200,000*l.* among 7 children. What will be the share of each ?

2. How much a day is 500,000*l.* a year ?

3. Forty tons of flour is divided among 30,000 men. How much will each get ?

4. A gross of small castings weighs 9 tons 1 cwt. 1 qr. 4 lbs. What is the weight of each separately?

5. A mile of telegraph wire weighs a ton. What is the weight per foot?

6. Divide 37 acres 3 r. 30 poles into 22 equal lots.

7. A butt of sherry contains 108 gallons, and costs me 67*l.* 10*s.* How much does it stand me in a dozen?

8. How many yards of carpet, yard and a half wide, will cover a floor 45 feet by 75? How much will it cost at 5*s.* 6*d.* a yard run?

9. Telegraph poles are placed 132 feet apart. How many will be wanted for 167 miles?

10. Certain castings weigh 28 lbs. 9 oz each. How many go to the ton, and how many to 173 tons 5 cwt.?

11. How many miles of wire, 25 feet to the ounce, can be drawn from 1 cwt. of metal?

12. How many rails 27 feet 6 in. long are required per mile for a double line of railway?

13. There are 625 steps in an ascent of 296 feet, measured vertically. What is the height of the risers in inches and eighths?

14. What must be the circumference of a gig-wheel which makes 6000 revolutions in 17 miles?

15. A gentleman tests a pedometer by walking 7½ miles, during which he finds it has scored 8347 double paces. What is his single pace?

16. The rental of a parish is 14,350*l.* and the land is let at 4*l.* 3*s.* 6*d.* per acre. What is the acreage?

17. There are 1435 telegraph poles in a distance of 60 miles. How far are they apart?

What assumption do you make in working this sum by division?

18. For a voyage of 17 weeks I take provisions, &c., to the amount of 48 tons 4 cwt. 2 qrs. 20 lbs. 9 oz. Supposing the ship's company to consist of 73 men, how much does this allow per man per day?

19. On a certain parallel the earth measures 12,315 miles round. What is the length of a minute of longitude on that parallel?

20. A rectangular tank 60 feet by 10 feet by 7 feet 6 in. deep is drawn off in square buckets each $9 \times 9 \times 8$ inches. How many buckets are there to be filled?

CHAPTER VIII.

FRACTIONS.—PART I.

A FRACTION of anything is a given portion of it. If we divide anything, no matter how, any piece of it is a fraction of it.

The simplest fraction is *one-half*, and is expressed as follows: $\frac{1}{2}$.

If we divide a thing, say a cake, into three equal pieces, and take one of them, we call that piece *one-third* of the cake, and we express it as a fraction thus—$\frac{1}{3}$; and so, if we divide it into four, and take one portion, we call it a quarter, or *one-fourth*, and write it $\frac{1}{4}$. So also, if we divide the cake equally among 14 children, each child gets one-fourteenth, or $\frac{1}{14}$.[1]

It is evident that the whole cake may be considered as two halves (which we write $\frac{2}{2}$), three thirds ($\frac{3}{3}$), four quarters ($\frac{4}{4}$), and the same with any other number; for example, fifteen fifteenths ($\frac{15}{15}$) is the whole cake. Again, since we cannot divide a cake into less than two pieces, it is clear that division by the number *one* leaves it just as it was, one whole

[1] If we divide a cake into x pieces, we should call each piece $\frac{1}{x}$; if into a pieces, each would be $\frac{1}{a}$.

cake, which we may therefore write $\frac{1}{1}$, and this is all we mean by writing

$$1=\frac{1}{1}=\frac{2}{2}=\frac{3}{3}=\frac{4}{4}=\frac{14}{14}=\frac{15}{15}=\frac{37}{37}, \text{ \&c. \&c.}=\frac{1001}{1001}.$$[1]

If we divide the cake into a good many parts (say 12), and take any number of these parts (say 7), we express the whole portion taken thus : $\frac{7}{12}$. The number below the line is called the denominator, and expresses the number of parts into which we divide the thing ; the number above the line is called the numerator, and expresses the number of parts taken :—

$$\text{Fraction} = \frac{\text{numerator}}{\text{denominator}}$$

Thus sevenpence is $\frac{7}{12}$ of a shilling—a penny being one-twelfth part of a shilling, and seven of them being taken. It is obvious that this is the same result as we should get by dividing 7 shillings among 12 persons.[2] For there are 84 pence altogether, and they would each get sevenpence.

If we divide a cake into, say, *twelve* parts, and take six of them, it is clear that we take *half* the cake ; hence, one-half $= \frac{\text{six}}{\text{twelve}}$, or $\frac{1}{2}=\frac{6}{12}$.

Now 6 is six times 1, and 12 is six times 2 ; so that we do not alter the value of the fraction $\frac{1}{2}$ by multiplying numerator and denominator by *six*. If we suppose the thing divides into *ten* parts, and *five* of them taken, we shall see that

[1] So, generally, $1 = \frac{x}{x}$, or $= \frac{a}{a}$. This may be verified in any given case, except when x or a is made = zero. The expression $\frac{0}{0}$ has no definite meaning, and it is easily seen that this is the only case in which we cannot verify the rule $\frac{x}{x} = 1$.

[2] And, generally, if we divide n things among d persons, each person gets $\frac{n}{d}$; the share being just what he would get if we divided one thing into d parts, and gave him n of them.

$\frac{1}{2}=\frac{5}{10}$, or $\frac{1}{2}$ is not altered by multiplying numerator and denominator by 5. We may find, upon trial, that the fraction $\frac{1}{2}$ is not altered in value by multiplying both numerator and denominator by the same number, be that number what it may : thus—

$$\frac{1}{2} = \frac{73 \times 1}{73 \times 2} = \frac{73}{146}; \frac{1}{2} \quad \frac{14 \times 1}{14 \times 2} = \frac{14}{28}, \text{\&c.}$$

Moreover, if we take *four* parts out of twelve, we shall have taken $\frac{1}{3}$ of the whole. This may be best seen by dividing the cake first into three equal parts, and then each of these into quarters. There will of course be 12 equal parts altogether. Hence—

$$\frac{1}{3} = \frac{4}{12}; \text{ but } \frac{4}{12} = \frac{4 \times 1}{4 \times 3}$$

or the fraction $\frac{1}{3}$ is not altered by multiplying numerator and denominator by *four* (or, as may be found by trial in a similar way, by any other number). It is an obvious extension of this rule that NO FRACTION IS ALTERED IN VALUE BY MULTIPLYING THE NUMERATOR AND DENOMINATOR BY THE SAME NUMBER, whatever that number may be.

This rule will be found to hold in every case in which we can take the trouble and find the means to verify it. It is the most important principle of fractions, and we shall hereafter find that the whole doctrine of proportion depends upon it.[1]

[1] Expressing it in letters, we write the rule thus :—

$$\frac{n}{d} = \frac{n \times x}{d \times x} = \frac{n \times y}{d \times y}, \text{ whatever be } x \text{ and } y\,;$$

provided x and y are not zero.

This principle that a fraction is not altered by multiplying or dividing top and bottom by the same multiplier, is to be understood of finite quantities only. The multiplier or divisor must not be zero or infinity. The reasoning which we have used depends upon the verification, which cannot be applied in such a case, and our applying it to zero or infinity would land us in absurd conclusions. Thus, $\frac{0}{0}$ is not necessarily unity,

For example, $\frac{3}{7} = \frac{3 \times 2}{7 \times 2} = \frac{6}{14}$; $\frac{3}{7} = \frac{3 \times 11}{7 \times 11} = \frac{33}{77}$.

But this is not all; we have noticed that the fraction $\frac{5}{10}$ is the same as the fraction $\frac{1}{2}$, and therefore is not altered if we *divide* numerator and denominator by 5. So with $\frac{6}{12}$ if we divide by six. This leads us to the general law, that if we divide numerator and denominator by the same number, we do not alter the fraction, whatever the number may be;[1] and we shall find this to be true whenever we have an opportunity of testing it. Thus—

$$\frac{5}{10} = \frac{\frac{5}{5}}{\frac{10}{5}} = \frac{1}{2};\ \frac{6}{12} = \frac{\frac{6}{6}}{\frac{12}{6}} = \frac{1}{2}; \text{ and also}$$

$$\frac{\frac{21}{7}}{\frac{81}{9}} = \frac{3}{9} = \frac{1}{3};\ \frac{\frac{22}{11}}{\frac{39}{3}} = \frac{2}{13}$$

It is customary to call fractions in which the denominator exceeds the numerator *proper fractions*. It will be noticed that all such fractions are less than unity. Thus $\frac{7}{8}$, $\frac{3}{5}$ and $\frac{38}{39}$ are all proper fractions, and are all less than unity. For unity may be written as $\frac{8}{8}$, $\frac{5}{5}$, or $\frac{39}{39}$, and it is clear that $\frac{7}{8}$ is less than $\frac{8}{8}$, $\frac{3}{5}$ less than $\frac{5}{5}$, $\frac{38}{39}$ less than $\frac{39}{39}$, and so on.

Where the numerator exceeds the denominator, the fraction is called an *improper fraction*. The distinction is not so material as the name would imply; for, practically, proper and improper fractions are *worked* as if there were no difference between them. As examples of improper fractions we may take $\frac{9}{8}$, $\frac{7}{5}$, and $\frac{39}{38}$. It is easy to see that $\frac{9}{8}$ is $\frac{8}{8}$ and $\frac{1}{8}$, or one and one-eighth (and easier still to see that $\frac{5}{4}$ or five

and there is nothing in arithmetic to show us what it does mean. We must therefore not rely upon any reasoning which implies that a specific value attaches to this expression.

[1] Expressing this rule in letters, we say:—

$$\frac{\frac{x}{x}}{\frac{d}{x}} = \frac{n}{d}, \text{ whatever be the value of } x, \text{ other than zero.}$$

quarters is $1\frac{1}{4}$, or one and a quarter). So *every* improper fraction can be reduced into a whole number *plus* a fraction. You have already had a hint of this in the way in which you have been taught to do *division*. For when you do such a sum as this—

$$
\begin{array}{l}
359)\ 11975\ (33\frac{128}{359} \\
\quad\ \ \underline{1077} \\
\quad\ \ \ 1205 \\
\quad\ \ \ 1077 \\
\quad\ \ \ \ 128
\end{array}
$$

you have simply chosen another way of expressing that $\frac{11975}{359}$ is 33 *and* $\frac{128}{359}$.

Perhaps you may have by this time anticipated the rule for converting an improper fraction into a *mixed number*. Divide numerator by denominator as in common division; the *quotient* is the whole number, and the *remainder* is the numerator of the proper fractions which accompanies it: thus, taking the previous example—

$$\frac{11975}{359} \text{ is } 33\frac{128}{359}$$

Multiplication of a Fraction by a Whole Number.

The denominator of a fraction merely indicates the number of parts into which the unit is divided. The numerator indicates how many of these parts are taken. If, for the moment, we regard each *part* as a unit, it is easy to see that we can multiply the whole thing by simply multiplying the number of parts taken. That is to say, the direct way of multiplying a fraction is simply to multiply its numerator and to leave its denominator unchanged. For instance, 3 times $\frac{3}{5}$ is $\frac{9}{5}$.

Division of a Fraction by a Whole Number.

From what we have said it is clear that the direct way is simply to divide the numerator. Thus to divide $\frac{12}{7}$ by 4, we

have simply to take one quarter of 12 sevenths, or 3 sevenths, which we write as $\frac{3}{7}$.

Another way of performing the same operation is to leave the numerator unchanged, and to multiply the denominator by the divisor. For consider $\frac{7}{12}$ of a shilling, which is 7 pence. If we divide each penny into four farthings, and take 7 of these farthings, it is clear that we shall have $\frac{1}{4}$ of the 7 pence. But these 7 farthings are each of them $\frac{1}{48}$ of a shilling, and altogether are $\frac{7}{48}$, that is to say, $\frac{7}{12 \times 4}$. Now the reason given in this illustration will apply to every conceivable case, except that we cannot always give specific names, like pence and farthings, to the fractions we use. Clearly that does not affect the reasoning.

To Multiply by a Fraction.

Let us again consider 7 pence; and suppose that we want to multiply it by $\frac{3}{4}$ or, what is the same, to take three quarters of it. If we take 3 farthings out of every penny, it is clear that we shall do this, and we shall get 7×3 farthings. But a shilling is 4×12 farthings; therefore, the result we have obtained is $\frac{3 \times 7}{4 \times 12}$ of a shilling. Hence we see that to multiply $\frac{7}{12}$ by $\frac{3}{4}$ we must multiply the 7 by 3 and the 12 by 4. The reasoning is evidently general, and hence we get the rule that to multiply two fractions we must multiply the numerators together for a new numerator, and the denominators together for a new denominator. Thus $\frac{5}{6} \times \frac{7}{8} = \frac{35}{48}$; $\frac{5}{6} \times \frac{6}{7} = \frac{5 \times 6}{6 \times 7} = \frac{5}{7}$, since we may divide out the 6.

A fraction of a fraction implies the multiplication of the two fractions. For instance, $\frac{2}{3}$ of $\frac{4}{7}$ is the same as $\frac{2}{3} \times \frac{4}{7}$, and $\frac{3}{5}$ of $\frac{20}{17}$ is $\frac{3}{5} \times \frac{20}{17} = \frac{12}{17}$.

The truth of this is easily seen by considering an example :—$\frac{2}{3}$ of any unit is evidently the same as that unit $\times$ $\frac{2}{3}$. Now, if that unit be a fraction of something else, say $\frac{9}{10}$, it is clear that our result will be $\frac{2}{3} \times \frac{9}{10}$.

Division by a Fraction.

This at first sight is puzzling, but we can give a meaning to it as follows :—Consider for a moment what is meant by such an expression as the following: 'Divide a shilling by a farthing.' It can receive no other interpretation than that of being a direction to find out how many times a farthing is contained in a shilling. If I divide a shilling by a penny, I get 12. If I divide a shilling by a farthing, I get 48; that is to say, dividing by a quarter of a penny gives a result four times as great as dividing by one penny; in other words, dividing by $\frac{1}{4}$ is the same as multiplying by 4. Now suppose that, instead of dividing by $\frac{1}{4}$, we want to divide by $\frac{3}{4}$. Since the divisor is three times as great, the quotient will be only $\frac{1}{3}$, and thus the effect of dividing by $\frac{3}{4}$ is equivalent to multiplying by $\frac{4}{3}$. This reasoning again is perfectly general, and it is possible to show in every case that can be conceived, that to divide by a fraction is equivalent to turning that fraction upside down and multiplying. Thus—

$$\frac{9}{10} \div \frac{7}{8} = \frac{9}{10} \times \frac{8}{7} = \frac{9 \times 8}{10 \times 7} = \frac{9 \times 4}{5 \times 7} = \frac{36}{35} = 1\frac{1}{35}$$

It is possible in the same way to get at once at the result of multiplying and dividing several sets of fractions. Thus, if we have to divide $\frac{9}{10}$ by $\frac{7}{8}$, to divide that result by $\frac{4}{3}$, to multiply that result by $\frac{5}{6}$, and then by $\frac{5}{4}$, we have simply to invert all the divisors, and then multiply all the fractions so changed. We obtain in this way—

$$\frac{9}{10} \times \frac{8}{7} \times \frac{3}{4} \times \frac{5}{6} \times \frac{5}{4} = \frac{9 \times 8 \times 3 \times 5 \times 5}{10 \times 7 \times 4 \times 6 \times 4} = \frac{5400}{6720}.$$

This fraction may be reduced by striking out some of the multipliers which occur in both numerator and denominator. Thus, the 8 and one of the 5's in the numerator divide

out with the 10 and one of the 4's in the denominator, leaving

$$\frac{9 \times 3 \times 5}{7 \times 6 \times 4}$$

Again, the 3 divides out with the 6 below it, leaving

$$\frac{9 \times 5}{7 \times 2 \times 4} = \frac{45}{56}$$

Improper fractions or mixed fractions present no further difficulty than proper fractions, except that mixed numbers must first be reduced to simple fractions. For instance, suppose we have to divide $3\frac{3}{4}$ by $2\frac{1}{4}$. Reducing both of these to simple fractions, the operation to be performed is—

$$\frac{15}{4} \div \frac{9}{4} = \frac{15 \times 4}{4 \times 9} = \frac{15}{9} = \frac{5}{3} = 1\frac{2}{3}.$$

The simplification of fractions cannot be effected with completeness and certainty without a knowledge of the rules of *Greatest Common Measure* and *Least Common Multiple.* Meanwhile many of the reductions can be done at sight by the exercise of a little discrimination.

We shall give the rules for the *Greatest Common Measure* and the *Least Common Multiple* in a later chapter, and we shall then give some further explanations of the simplification of fractions, and of the subject-matter of the following section.

On the Comparison, Addition, and Subtraction of Simple Fractions.

If two fractions have the same denominator, we can compare them, add them, or subtract them, by comparing, adding, or subtracting their numerators. For example, if we want to add $\frac{7}{12}$ and $\frac{4}{12}$ of a shilling, we have only to work

in pence to see that we get $\frac{11}{12}$ of a shilling; and so if we wanted to subtract one from the other, we should get $\frac{3}{12}$ of a shilling; and this principle is evidently true whether we happen or not to have a specific name for the division indicated by the denominator, as in the case of a penny for $\frac{1}{12}$ of a shilling. Hence if we know how to reduce fractions to others having a common denominator, we can always add or subtract them, or compare them with one another.

This operation of reduction to a common denominator is necessary. For instance, we cannot say that $\frac{3}{5}$ and $\frac{4}{7}$ make 7 of anything. But we *can* say that $\frac{21}{35}$ and $\frac{20}{35}$ are $\frac{41}{35}$ of anything whatever.

To Reduce Fractions to a Common Denominator.—For this purpose it is necessary to find some number which can be divided by all the denominators, and to make this number serve as a common denominator. In order that the value of the fractions may not be changed, the numerator of each fraction must be multiplied by the number of times the denominator of that fraction is contained in the common denominator. For instance, in the case of $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$, it is clear that 2, 3, and 4 will all divide 12. Hence we may make 12 the common denominator. To reduce any fraction to one which will have 12 instead of 2 for its denominator, we have simply to multiply it by $\frac{6}{6}$, since $\frac{12}{2}=6$. In this way we see that $\frac{1}{2}=\frac{1}{2}\times\frac{6}{6}=\frac{6}{12}$. In the same way, $\frac{1}{3}=\frac{1}{3}\times\frac{4}{4}=\frac{4}{12}$. Again, $\frac{1}{4}=\frac{1}{4}\times\frac{3}{3}=\frac{3}{12}$. We are now in a position to add these fractions, or to subtract them, in any order we please. For instance, $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=\frac{6}{12}+\frac{4}{12}+\frac{3}{12}=\frac{13}{12}=1\frac{1}{12}$;

$$\frac{1}{2}-\frac{1}{3}+\frac{1}{4}=\frac{6}{12}-\frac{4}{12}+\frac{3}{12}=\frac{5}{12}$$

In the case before us it was easy to see that 12 would suit our purpose; but it is not always so obvious what is the least number we can use for the common denominator. That will be explained in the chapter on the Least Common Multiple; meanwhile it may be remarked that the continued product of all the different denominators will always answer

the purpose. For instance, in the case we have taken, the continued product of the denominators is $2 \times 3 \times 4 = 24$, and we might perform the addition as follows: $\frac{1}{2}+\frac{1}{3}+\frac{1}{4} = \frac{12}{24}+\frac{8}{24}+\frac{6}{24}=\frac{26}{24}=1\frac{2}{24}=1\frac{1}{12}$ as before.

Examples in Fractions. (Part I.)

1. Reduce to improper fractions :—

(1) $3\frac{1}{7}$. (2) $1\frac{4}{9}$. (3) $2\frac{11}{31}$.
(4) $7\frac{5}{9}$. (5) $18\frac{7}{45}$. (6) $69\frac{61}{71}$.
(7) $993\frac{343}{729}$. (8) $192\frac{223}{1001}$. (9) $91\frac{779}{1441}$.
(10) $100\frac{99}{100}$. (11) $99\frac{99}{101}$. (12) $1001\frac{999}{1001}$
(13) $12345679\frac{8}{9}$. (14) $77\frac{12}{13}$. (15) $91\frac{10}{11}$.
(16) $143\frac{6}{7}$. (17) $111111\frac{1}{9}$. (18) $10101\frac{1}{99}$.

2. Reduce to whole or mixed numbers :—

(1) $\frac{1002}{7}$. (2) $\frac{69}{17}$. (3) $\frac{68}{19}$.
(4) $\frac{431}{69}$. (5) $\frac{1001}{77}$. (6) $\frac{8029}{2738}$.
(7) $\frac{7168}{1024}$. (8) $\frac{1484}{621}$. (9) $\frac{744829}{5953}$.
(10) $\frac{2658683}{743027}$. (11) $\frac{3124769}{1041589}$. (12) $\frac{83527941}{41283404}$
(13) $\frac{87746631}{14388471}$. (14) $\frac{2374283}{1483}$. (15) $\frac{4744283}{1483}$.
(16) $\frac{185534257}{79432563}$.

3. Reduce the following numbers to fractions having a given denominator :—

(1) 2 to denrs. 5 and 31 (2) 7 to denrs. 11, 13 and 143.
(3) 143 to denr. 11. (4) 9 to denr. 11.
(5) 909 to denrs. 11 and 101. (6) 101 to denrs. 9 and 99.

4.

(1) $\frac{3}{4} \times 6$. (2) $\frac{5}{8} \times 5$.
(3) $\frac{3}{5} \times 10$. (4) $\frac{13}{77} \times 1001$.
(5) $\frac{137}{10001} \times 146$. (6) $\frac{1001}{10001} \times 137 \times 365$.
(7) $\frac{10001}{1002001} \times 7 \times 7 \times 11 \times 11 \times 13 \times 13 \times 17 \times 17$.

5.

(1) $\frac{3}{4} \div 6$.
(2) $\frac{5}{8} \div 5$.
(3) $\frac{8}{5} \div 10$.
(4) $\frac{13}{77} \div 1001$.
(5) $\frac{107}{10301} \div 146$.
(6) $\frac{10001}{137} \div 146$.
(7) $\frac{77}{13} \div 1001$.
(8) $\frac{1001}{13} \div 77$.
(9) $\frac{7 \times 7 \times 11 \times 11 \times 13 \times 13 \times 17 \times 17}{1002001} \div 1001$.
(10) $\frac{3682778}{1001} \div 5117$.

6.

(1) $\frac{3}{4}$ of 2.
(2) $\frac{9}{7}$ of 14.
(3) $21 \times \frac{5}{7}$.
(4) $\frac{73}{137}$ of 10001.
(5) $\frac{137}{73}$ of 10001.
(6) $100001 \times \frac{9090}{9091}$.
(7) $1000001 \times \frac{9900}{9901}$.

7.

(1) $\frac{3}{4}$ of 27*l.* 14*s.*
(2) $\frac{5}{7}$ of 31*l.* 10*s.*
(3) $\frac{9}{14}$ of 45*l.* 3*s.*
(4) $\frac{3}{7}$ of 1 ton 15 cwts.
(5) $\frac{7}{16}$ of 3 tons.
(6) $\frac{5}{64}$ of 4 tons 16 cwts.
(7) $\frac{7}{128}$ of 3 tons 15 cwts.
(8) $\frac{5}{77}$ of $3\frac{1}{2}$ statute miles.
(9) $\frac{47}{121}$ of a square mile.
(10) $\frac{1000000000}{1003003001}$ of 8288181 square miles.
(11) $\frac{13}{4032}$ of 4 lbs. 5 oz. troy.
(12) $\frac{567}{876}$ of 1 ton 14 cwts. 3 qrs.

8.

(1) $\frac{5}{7} \times \frac{21}{43}$.
(2) $\frac{3}{8} \times \frac{11}{9}$.
(3) $\frac{4}{9} \times \frac{27}{16}$.
(4) $1\frac{4}{9} \times \frac{13}{77}$.
(5) $1\frac{4}{7} \times 1\frac{1}{13}$.
(6) $12\frac{29}{81} \times 8\frac{1}{91}$.
(7) $\frac{11}{365} \times 42\frac{219}{1183}$.
(8) $1\frac{2}{9} \times 1\frac{2}{11} \times 1\frac{4}{17} \times 1\frac{2}{19}$.
(9) $1\frac{8}{29} \times 1\frac{6}{37} \times 1\frac{4}{43} \times 1\frac{6}{47}$.
(10) $2\frac{1}{19} \times 2\frac{6}{19} \times 2\frac{12}{37} \times 1\frac{33}{43}$.

9.

(1) $\frac{7}{8} \div \frac{1}{10}$.
(2) $\frac{11}{7} \div 1\frac{1}{10}$.
(3) $\frac{14}{5} \div 2\frac{1}{10}$.
(4) $1001 \div \frac{77}{13}$.
(5) $10001 \div \frac{137}{365}$.
(6) $10001 \div \frac{365}{137}$.
(7) $\frac{31}{28} \times \frac{44}{39} \div \frac{3}{91}$.
(8) $\frac{19}{187} \times \frac{17}{341} \div 1\frac{6}{11}$.
(9) Divide $2\frac{1}{19} \times 2\frac{6}{19} \times 2\frac{12}{37} \times 1\frac{33}{43}$ by $1\frac{2}{9} \times 1\frac{2}{11} \times 1\frac{4}{17} \times 1\frac{2}{19}$.

10. Divide the product of $\frac{1}{73}$ and $\frac{1}{137}$ by the product of $\frac{1}{7} \times \frac{1}{11} \times \frac{1}{13}$.

11. Find which is the greatest fraction in each of the following pairs :—

(1) $\frac{5}{7}$ and $\frac{8}{11}$. (2) $\frac{6}{11}$ and $\frac{13}{28}$.
(3) $\frac{36}{77}$ and $\frac{29}{62}$. (4) $\frac{34}{45}$ and $\frac{59}{85}$.

12. Arrange the following fractions in order of magnitude :—

(1) $\frac{19}{25}$, $\frac{3}{4}$, $\frac{16}{21}$, $\frac{13}{17}$, $\frac{10}{13}$, $\frac{7}{9}$, $\frac{4}{5}$, $\frac{22}{29}$, $\frac{25}{33}$, $\frac{28}{37}$.
(2) $\frac{8}{25}$, $\frac{9}{28}$, $\frac{17}{53}$, $\frac{26}{81}$, $\frac{25}{78}$, $\frac{28}{87}$.

13. It is a general rule that if we add the numerators of a number of fractions, and also the denominators, the fraction given by $\frac{\text{sum of numerators}}{\text{sum of denominators}}$ lies between the greatest and least of the original fractions. Verify this on the previous examples; for instance—12 (2)—

$$\frac{8 + 9 + 17 + 26 + 25 + 28}{25 + 28 + 53 + 81 + 78 + 87}$$

lies between the greatest and least of the separate fractions named in that example. Set some other examples for yourself on this point : thus, $\frac{3 + 5}{8 + 7}$ lies between $\frac{3}{8}$ and $\frac{5}{7}$, and also between $\frac{3}{7}$ and $\frac{5}{8}$.

14. Find the value of :—

(1) $\frac{3}{4}+\frac{5}{6}$. (2) $\frac{13}{15}+\frac{17}{20}$. (3) $\frac{17}{19}+\frac{19}{17}$.
(4) $\frac{18}{35}+\frac{17}{70}$. (5) $\frac{19}{25}+\frac{17}{100}$. (6) $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$.
(7) $\frac{1}{2}-\frac{1}{3}+\frac{1}{4}$. (8) $\frac{1}{7}+\frac{1}{11}+\frac{1}{13}$. (9) $\frac{1}{7}-\frac{1}{11}+\frac{1}{13}$.
(10) $\frac{1}{7}+\frac{1}{11}-\frac{1}{13}$. (11) $\frac{1}{7}+\frac{1}{11}+\frac{1}{13}$. (12) $\frac{1}{73}-\frac{1}{137}$.
(13) $\frac{19}{35}+\frac{17}{100}-\frac{13}{70}$. (14) $\frac{25}{17}-\frac{18}{19}-\frac{1}{323}$.
(15) $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}$.
(16) $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}$.

CHAPTER IX.

PROPORTION AND RATIO.

PROPORTION is one of those ideas which cannot be communicated by mere definition. It naturally arises out of the attempt to compare the *relative size* of two magnitudes of the same description without reference to their *absolute size*. Thus the idea of the half or the double includes the idea of a particular proportion; for in saying that one thing is half or double another, we do not concern ourselves with the *actual* size of either, or, indeed, with any idea of actual size at all, at the same time that we express completely and accurately the relation between the two sizes. Moreover this comparison is of the nature of multiplication or division, and not of addition or subtraction. For instance, if I compare the lengths of two sticks by subtraction, and say that one stick is 1 foot longer than the other, I make use of the absolute measure, 1 foot. But if I want to get rid of the idea of absolute measure altogether, I must say that one stick is so many times as long as the other; the 'so many' times being whole or fractional.

When we consider two quantities only, we call this relation RATIO. It has no definite meaning unless the quantities are of the same kind. Thus the ratio of twelve things to six things of the same sort is definite, meaning twice as many; but the ratio of twelve sheep to six oxen is meaningless, because we do not know how to compare a sheep and an ox numerically. But if we can refer both to a standard of weight or value, we get rid of the difficulty, and make the question definite.

The simplest expression of *ratio* is that which exists between the two numbers. For instance, the simplest expression of double or half arises out of the comparison of the numbers 1 and 2, considered as mere numbers, and without reference to what the unit 1 expresses.

The reference to a common standard, alluded to above, involves the *comparison of ratios*. This is called PROPORTION. A ratio is itself the comparison of two things. Proportion, therefore, or the comparison of ratios, involves generally the comparison of four things, and requires at least three things to be compared. The comparison chiefly useful is the *equality of ratios*, and this it is which is commonly called PROPORTION.

Four quantities are said to be proportional when the ratio of the first to the second is the same as that of the third to the fourth. Simple cases of this are easily stated. For instance, the ratio of 4 to 1 is evidently the same as that of 12 to 3, because 12 is 4 times 3, just as 4 is 4 times 1; or we may invert the idea, and infer the proportion or the equality of the two ratios from the observation that 1 is $\frac{1}{4}$ of 4, just as 3 is $\frac{1}{4}$ of 12. A case a little more complex is when we assert that the ratio of 6 to 10 is the same as that of 9 to 15. For the number of times which 10 contains 6 is the same as the number of times which 15 contains 9, this number being in each case the fraction $1\frac{2}{3}$.

The ideas of numerical fractions and of ratios hang so closely together, that, for practical purposes, they may be considered as two views of the same thing. There is actually, however, this difference, that ratio does not necessarily imply the idea of partition as a fraction does, and may exist between two magnitudes neither of which is numerical. In fact a fraction implies one thing cut into pieces, either really or in imagination; while ratio compares two things without cutting them up.

Moreover, a fraction takes a restricted view of even definite ratio. Proportion may include the comparison of

ratios which are not definite, and which are incapable of being separately expressed as fractions.[1]

The ratio between two numbers is expressed by writing them one after the other with a couple of dots or a colon between, thus 3 : 5 ; and if we wish to express that this is the *same* ratio as that which holds between 6 and 10, we write down the two ratios with four dots between them, thus 3 : 5 :: 6 : 10 ; and we read it thus : 'As three is to five, so is six to ten.' If we express it fractionally, it is equivalent to the statement that $\frac{3}{5}=\frac{6}{10}$.

It will be noticed that one of these four quantities depends on the three others, that is to say, that if we name three of them the fourth is fixed. For instance, if we write—

$$\frac{3}{5}=\frac{6}{\text{something}}$$

the 'something' which forms the denominator of the fraction is not anything we please. In point of fact it is evident it can be nothing else than 10. The object of the rule of proportion or rule of three, as it is commonly called, is, when three terms of a proportion are given, to find the fourth. The rule is this :—

[1] I do not think it necessary to restrict the idea of proportion to numerical ratio. In fact, it appears to me to be a consequence of

3 shillings : 5 shillings :: 6 yards : 10 yards,

that

3 shillings : 6 yards :: 5 shillings : 10 yards ;

and that while I do not know what exactly is the meaning of the expression

3 shillings : 6 yards

standing by itself, I am enabled to assert that there is a connection between 3 shillings and 6 yards, subject to the rules of proportion, which is the same as the connection between 5 shillings and 10 yards, whatever that may be.

To deny this appears to me to imply an objection to concrete arithmetic generally. I do not assert that there is a definite ratio between them, but merely a relation, which we may make definite if we please, and which we can always treat as if it were definite.

If four quantities are proportional, the product of the first and fourth is equal to the product of the second and third.

Or as this is sometimes written :—

The product of the extremes is equal to the product of the means.

For instance, let $a : b :: c : d$; then if we express these ratios as fractions we have $\frac{a}{b} = \frac{c}{d}$. Now, if two things are equal, and we multiply each of them by any definite quantity, the products will be equal also. Let us apply this principle to the two equal fractions $\frac{a}{b}$ and $\frac{c}{d}$, and multiply each of them by the product $b \times d$, we get $\frac{a \times b \times d}{b} = \frac{c \times b \times d}{d}$. Now $\frac{a \times b \times d}{b}$ is $a \times d$, and $\frac{c \times b \times d}{d}$ is $c \times b$. Hence finally we obtain—

$$a \times d = c \times b,$$

and this equality, it must be remembered, we have obtained *as a mere consequence* of the proportion $a : b :: c : d$.

In the example that we started from, viz. $3 : 5 :: 6 : 10$ this is easily verified, for 3×10, and 5×6 are each of them 30.

The equality $a \times d = b \times c$ enables us to find any one term of a proportion from the three others. For if two equal quantities be divided by any one definite quantity, it is clear that the quotients will be equal. Now it being known that $a \times d = c \times b$, suppose we want to find a, we have merely to divide each of these quantities by d, and as the quotients will still be equal, we shall have $a = \frac{c \times b}{d}$. If we wanted to find d, we should simply have to divide by a, obtaining $d = \frac{c \times b}{a}$. So also if we wished to obtain b or c we should have had to divide by c or b.

To recapitulate all this :—

If $a : b :: c : d$, it follows, *firstly*, that $a \times d = b \times c$.

And *secondly* that—

$$= \frac{b \times c}{d}, \quad d = \frac{b \times c}{a}, \quad b = \frac{a \times d}{c}, \quad c = \frac{a \times d}{b}$$

A familiar example is to be found in such a case as this. If 3 yards of stuff cost a given price, what will 7 yards cost? The first question which we must put to ourselves is this :—May we take it for granted that the prices are proportionate to the lengths? If we may, we may at once write down the proportion—

3 : 7 :: price given : price required.

and then by what went before we have :—

$$\text{price required} = \frac{7 \text{ times price given}}{3}.$$

If, for instance, the 3 yards cost 13 shillings, the price required will be $\frac{7 \times 13 \text{ shillings}}{3}$, or $\frac{91 \text{ shillings}}{3}$, or 30 shillings and 4 pence.

The question whether the quantities may be assumed to be proportional is one never to be lost sight of. If the last example referred to ribbons or ropes which are sold *by length*, as it is called, it would fall within the rule; but if the question had been about ladders, we could not have applied the rule, because a long ladder requires to be stronger than a short ladder, and therefore weighs more and costs more per foot run. In other words, the prices of two ladders are not proportionate to their lengths, and therefore the rule of three must not be applied to them.

I shall now discuss some examples.

1. If 14 yards of cloth cost 7*l.* 5*s.* 10*d.*, what will 55 cost? Here we evidently have—

					£	s.	d.		
14	:	55	::		7	5	10	:	answer.

20

145

12

1750 pence

55

8750

8750

7) 96250

2) 13750

12) 6875 answer in pence

20) 57,2 11

£28 12*s.* 11*d.* *Answer.*

The first step is to reduce the third term to pence. We then multiply by the second and divide by the first. This gives the fourth term in *pence*, which we afterwards reduce.

We might do it without reduction, as follows :—

					£	s.	d.	
14	:	55	::		7	5	10	:
							5	
					36	9	2	
							11	
				2)	401	0	10	
				7)	200	10	5	
					28	12	11	*Answer.*

2. If $6\frac{1}{4}$ yards of stuff cost 3*s.* $7\frac{1}{2}$*d.*, what will $25\frac{1}{4}$ cost?

$6\frac{1}{4}$: $25\frac{1}{4}$ :: 3*s.* $7\frac{1}{2}$*d.*

Since a ratio is not altered by multiplying both its terms by the same quantity, we may get rid of the fractions by multiplying the first two terms by 4. We must not

multiply *all three* terms, because this would give us four times the answer. We get—

$$25 : 101 :: 3\text{s. } 7\tfrac{1}{2}\text{d.} :$$

$= 43\frac{1}{2}$ pence
$= 87$ halfpence
101

5) 8787 (2

5) 1757 (4

Answer $351\frac{12}{25}$ halfpence
$= 155\frac{37}{50}$ pence
$= 14s.\ 7\frac{3}{4}d.$ very nearly.

3. Reduce 1400 lbs. troy to avoirdupois weight. A troy pound is 5,760 grains, and an avoirdupois pound is 7,000 grains. Since the weights are to be the same, the number of pounds multiplied by the weight of a pound must be equal in either measure. That is—

$$1400 \times 5760 = 7000 \times \text{answer.}$$

Divide both sides by 7,000 and we get—

$$\text{Answer} = \frac{1400 \times 5760}{7000} = 2 \times 576$$

$$= 1152 \text{ avoirdupois pounds.}$$

This might have been stated as a proportion thus :—

$$7000 : 5760 :: 1400 : \text{answer.}$$

And this is what the old writers actually did. But it is more natural to state it as I have done, and the work is the same in both cases.

4. If a man can get through a piece of work by working 8 hours a day for 90 days, how long will it take him if he works 10 hours a day?

Here again the number of hours has to be the same in both cases, and we get—

$$8 \times 90 = 10 \times \text{answer}.$$
$$8 \times 9 = \text{answer} = 72.$$

5. The duty on tea being 6*d.* per lb. gives a revenue in a particular district of 1,500*l.* What will the revenue be if the duty is raised to $10\frac{1}{2}$*d.*?

This does not necessarily fall within the rule; for it may happen that raising the duty will diminish the quantity imported. Supposing, however, the quantity imported to be unaltered, we shall have—

6*d.* : $10\frac{1}{2}$*d.* :: 1500*l.* : answer.

And the answer is easily found to be 2,625*l.* But to get this answer it has been necessary to add to the question.

Of course, if any question is set as an example of any rule, we are at liberty to assume all that is necessary to apply the rule. But this principle is more useful in examinations than in practical questions.

Compound Proportion.

This arises out of such questions as the following. If 15 men working 8 hours a day can plough 240 acres of land in 10 days, how many days will it take 20 men working 10 hours a day to plough 600 acres? In such a question, the first thing is to get a distinct idea of the *supposition* as distinguished from the *question*. Here the supposition is, that 15 men plough 240 acres in 10 days of 8 hours. If, now, we arrange our question in the same way as the supposition it runs—20 men plough 600 acres in ? days of 10 hours: and the ? in the number of days is what we have to fill up.

If the terms of the *question* had been the same as the terms of supposition; that is, if the question had run—15 men plough 240 acres in ? days of 8 hours, it is clear that the number of days would have been unaltered, viz. 10. We

are now going to consider the separate effect of altering each term of the supposition to suit the question.

1st. Employing 20 men instead of 15 will evidently reduce the number of days in the proportion of 15 to 20; therefore, as far as that change alone is concerned the 10 days will have to be multiplied by the fraction $\frac{15}{20}$.

2nd. The number of acres is increased from 240 to 600. This will increase the number of days in the proportion of 600 to 240. The effect of that change upon the answer will therefore be to multiply it by $\frac{600}{240}$.

3rd. The alteration of the working day from 8 to 10 hours will reduce the number of days in the proportion of 10 to 8. Its effect, therefore, will be equivalent to multiplying by the fraction $\frac{8}{10}$.

We have thus found the separate effects of three changes, each of which acts independently of the other, and simply introduces its own multiplier. The conjoint effect upon the number of days will evidently be to multiply that number by—

$$\frac{15}{20} \times \frac{600}{240} \times \frac{8}{10}$$

that is to say, the answer will be—

$$\frac{15}{20} \times \frac{600}{240} \times \frac{8}{10} \times 10 \text{ days}$$
$$= \frac{3}{4} \times \frac{5}{2} \times 8 \text{ days} = 15 \text{ days.}$$

All questions of compound proportion are, in effect, solved in this way, and I am not aware that the principle can be more clearly put. That principle stated generally is, To find a multiplier, whole or fractional, which shall represent each separate change of each pair of terms in passing from the supposition to the question, and then to apply all these multipliers together to the odd term of the supposition. This will give the answer, which will thus fill up the blank in the question corresponding to the odd term of the supposition. For instance, what we have found by solving the foregoing example amounts to this—

If 15 men plough 240 acres in 10 days of 8 hours, *then*—
20 men plough 600 acres in 15 days of 10 hours.[1]

Again, if 1,500 navvies working 10 hours a day can cut a canal 20 miles long, 20 feet broad, and 6 feet deep, how many soldiers will it take to make a trench 10 miles long, 30 feet wide, and 5 feet deep, supposing that they only work 6 hours a day, and that in the same space of time 7 navvies can do the work of 10 soldiers?

The work may be arranged as follows:—

	Terms of Supposition	Terms of Question	Multiplier
men	1,500		
hours per day	10	6	$\frac{10}{6}$
length, miles	20	10	$\frac{10}{20}$
breadth, feet	20	30	$\frac{30}{20}$
depth, feet	6	5	$\frac{5}{6}$
soldiers instead of navvies. . .	10	7	$\frac{10}{7}$

Hence, the answer will be—

$$1500 \times \tfrac{10}{6} \times \tfrac{10}{20} \times \tfrac{30}{20} \times \tfrac{5}{6} \times \tfrac{10}{7} = 2232\tfrac{1}{7}.$$

That is to say, 2,233 soldiers will be required.

The last column is not strictly necessary, but I have used it for the sake of clearness. It is sufficient in practice to put a cross or other mark against that term of each pair which is to serve as numerator; and, indeed, with a little practice, the final fraction may be written down at once, without any previous arrangement in columns.

[1] The above statement includes 8 terms, of which if any 7 be given the 8th may be found from them. The student may therefore vary the questions as an exercise, by striking out any one of the 8 terms and finding it from the others. He should also be able to state the question in its altered form. Thus, supposing we strike out the 240 acres, and make that the term to be sought, the question will run—*If* 20 men plough 600 acres in 15 days of 10 hours each, how many acres will 15 men plough in 10 days of 8 hours?

Examples in Proportion.

1. What are a servant's wages for 219 days at 14*l.* 12*s.* a year?

2. A man's wages are 18*s.* a week; but he is paid quarterly. How much ought he to receive for the three months ending December 31?

3. How long must he work to earn 100*l.*?

4. A pig weighs 3 cwts.; what will it sell for at 3*s.* 2*d.* per 8 lbs.?

5. How much will 95 yards of paper cost at 3*s.* 6*d.* per piece of 11 yards?

6. Fifteen ounces of copper-ore are found to contain $1\frac{1}{4}$ oz. of pure copper. How much pure copper is there in a heap of 27 tons of such ore?

7. Water consists of eight parts by weight of oxygen and one of hydrogen. What weight of oxygen is there in 19 gallons of water?

8. In the common Gothic arch the height of the point above the capital is 13 fifteenths of the width. What will be the height for an arch of 65 feet in width?

9. A publisher sells to the trade 25 copies of a book for the marked or nominal price of 18 copies. How much does he actually receive for each copy of a 35*s.* book?

10. Shrubs are planted a yard apart upon two acres of ground. How much will they cost at 13*s.* 6*d.* per 1000?

11. How many yards of carpet three quarters of a yard wide will cover a room 27 feet square?

12. A market-woman buys herrings at 2*s.* 8*d.* the hundred of six-score and retails them at 4*s.* 6*d.* the true hundred. How many must she sell to earn half-a-sovereign?

13. 59 lbs. of a mixed metal are lowered into a full bucket of water, and the water which overflows is found to measure 195 cubic inches. What is the weight of a cubic foot of the metal, supposing a cubic foot of water to weigh $62\frac{1}{2}$ lbs.?

14. How many pairs of gloves at 1*s.* 10½*d.* a pair will cost as much as 141 yards of silk at 4*s.* 8*d.* a yard?

15. A fusible metal consists of three parts lead, five parts tin, and eight parts bismuth. How much of each will be wanted to make up 7 cwts. 3 qrs. of the alloy?

16. A degree of latitude in lat. 45° is 69 miles 121 feet, statute measure. What is the length of 1 second in inches?

17. A sidereal day is 23 h. 56 m. 4 sec. mean solar time. How many sidereal days are there in the sidereal year of 365¼ solar days?

18. Bar iron inch and a quarter square and 20 feet long is rolled down to ¾ inch square. How long will it be?

19. A plate of copper 2 feet long 9 inches wide and 2 inches thick is rolled into a sheet 3′ 6″ wide and 18′ long. How thick will it be?

20. A horse-power will lift 33,000 pounds one foot high per minute. How many gallons of water per day (of 24 hours) will an engine of 120 horse-power lift to the height of 140 fathoms?

21. Lead weighs $11\frac{4}{10}$ times, and platinum 21 times, as much as water. What weight of platinum will be equal in bulk to 112 lbs. of lead?

22. What bulk of lead will be equal in weight to 1,200 cubic inches of platinum?

23. Proof spirit contains 49 parts in every 100 of absolute alcohol. What is the quantity of absolute alcohol in 54 gallons of a mixture of 33 parts proof spirit and 14 parts water, neglecting the diminution of bulk which takes place in mixing spirit and water?

24. A garrison of 1,825 men consumed in 60 days 73 tons 9 cwt. 2 qrs. 16 lbs. of solid food. How much would be needed for a garrison of 15,000 men for half a year?

25. A falling body acquires a velocity of 32 feet per second. What velocity does it acquire in 3¾ seconds?

26. Supposing a body to fall 144 feet in 3 seconds, how long will it take to fall 1,600 feet?

27. A coach drives at the rate of $8\frac{1}{2}$ miles an hour; an hour and a quarter after it has started a horseman is sent after it who rides 14 miles an hour; how far off does he overtake it?

28. If $3\frac{3}{7}$ of a share cost 5*l.* what will $5\frac{5}{6}$ cost?

29. The weight of lead is $\frac{18}{35}$ and that of copper $\frac{29}{70}$ of an equal bulk of platinum. How many cubic inches of lead will be equal in weight to a cubic foot of copper?

30. In a quantity of mixed metal it is found that $\frac{13}{15}$ by weight consist of an alloy containing $\frac{37}{52}$ of nickel, while the remainder consists of $\frac{1}{4}$ copper and $\frac{3}{4}$ silver. the silver is extracted and found to weigh $2\frac{1}{3}$ lbs. troy. What is the weight of nickel in lbs. avoirdupois?

31. The quantity of saline matter in sea-water is $\frac{9}{250}$ in weight, and of this $\frac{11}{180}$ is magnesia. How many grains of magnesia are there in a cubic foot of sea-water, reckoned as 35 feet to the ton?

CHAPTER X.

PRACTICE.

THE object of Practice is to find the result of what may be done by compound multiplication by a shorter method than that rule affords. It is generally a combination of division and addition. Suppose, for example, we have to find the price of 3,789 articles at 6*s.* 8*d.* each. We can of course do this by multiplication and subsequent reduction, but since 6*s.* 8*d.* is $\frac{1}{3}$ of a pound it is evident that the price in pounds will be $\frac{1}{3} \times 3789$ or 1,263*l.* This is a simple case of what is called Practice.

The fractions used in Practice all have unity for their numerator, so that their use leads to simple division. Such fractions are called aliquot parts. Thus, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{7}$, &c., are aliquot parts, but $\frac{4}{5}$ or $\frac{3}{4}$ are not so.

Suppose then we have to find the cost of a number of articles at a certain price in shillings and pence. Our object is to split up this price into convenient portions each of which shall be an aliquot part either of the higher denomination (1*l.*) or of one another. We then find separately the cost as if each portion were the whole price, and then add them all together for the total cost. Thus, to find the cost of 2,791 articles @ 17*s.*; I observe that 10*s.* is $\frac{1}{2}$ of 1*l.*, that 5*s.* is $\frac{1}{2}$ of 10*s.*, and that 2*s.* is $\frac{1}{10}$ of 1*l.*, and also that 10 + 5 + 2 make up 17*s.* The sum is now done as follows:

10	$\frac{1}{2}$	2791	
5	$\frac{1}{2}$	1395	10
2	$\frac{1}{10}$	697	15
		279	2
		£2372	7

and the explanation of it is this:

If the articles cost 1*l.* each, the price will be 2,791*l.* The cost at 10*s.* each will be half this, and I therefore divide by 2 to get the cost at 10*s.* I again divide this by 2 to get the cost at 5*s.* In order to get the cost at 2*s.*, which is $\frac{1}{10}$th of 1*l.*, I go back to 2,791 and divide that by 10. I then add all these together for the result.

The remainders of the division must of course be reduced to shillings, and the remainders of the shillings, if any, to pence, and so on to farthings, if need be.

In the above example, if the price had been 1*l.* 17*s.*, the work would have been exactly the same, except that we should have added in the top line, thus getting 5,163*l.* 7*s.* for the total cost.

There are a great many little expedients in Practice, some of which we shall now proceed to explain, but it is only habit and frequent use which will give that readiness in seeing the shortest method which alone makes this rule advantageous in the business for which it is chiefly used, viz. bills and invoices, and certain statistics. The first thing is to be well acquainted with the aliquot parts of the pound or other unit; for the rule applies not only to money but to weights and measures.

The aliquot parts of a pound are—

10*s*. = $\frac{1}{2}$,	1*s*. 8*d*. = $\frac{1}{12}$,	6*d*. = $\frac{1}{40}$,	1½*d*. = $\frac{1}{160}$,
6*s*. 8*d*. = $\frac{1}{3}$,	1*s*. 4*d*. = $\frac{1}{15}$,	5*d*. = $\frac{1}{48}$,	1¼*d*. = $\frac{1}{192}$,
5*s*. = $\frac{1}{4}$,	1*s*. 3*d*. = $\frac{1}{16}$,	4*d*. = $\frac{1}{60}$,	1*d*. = $\frac{1}{240}$,
4*s*. = $\frac{1}{5}$,	1*s*. = $\frac{1}{20}$,	3*d*. = $\frac{1}{80}$,	¾*d*. = $\frac{1}{320}$,
3*s*. 4*d*. = $\frac{1}{6}$,	10*d*. = $\frac{1}{24}$,	3¾*d*. = $\frac{1}{64}$,	½*d*. = $\frac{1}{480}$,
2*s*. 6*d*. = $\frac{1}{8}$,	8*d*. = $\frac{1}{30}$,	2½*d*. = $\frac{1}{96}$,	and
2*s*. = $\frac{1}{10}$,	7½*d*. = $\frac{1}{32}$,	2*d*. = $\frac{1}{120}$,	¼*d*. = $\frac{1}{960}$.

The aliquot parts of a shilling are—

6*d*. = $\frac{1}{2}$,	3*d*. = $\frac{1}{4}$,	1½*d*. = $\frac{1}{8}$,	¾*d*. = $\frac{1}{16}$,	¼*d*. = $\frac{1}{48}$.
4*d*. = $\frac{1}{3}$,	2*d*. = $\frac{1}{6}$,	1*d*. = $\frac{1}{12}$,	½*d*. = $\frac{1}{24}$,	

The aliquot parts of a penny are ½*d*. and ¼*d*.

Our object is to use all these with the least possible trouble, and to avoid writing down any lines which are not to come in to the final addition. For instance, if we have 8.794 @ 2*s*. 0¼*d*., instead of taking 2*s*. as = $\frac{1}{10}$ of a £, and a farthing = $\frac{1}{96}$ of 2*s*., this last being an inconvenient divisor, we take it as 1*s*. 8*d*. = $\frac{1}{12}$. 4*d*. = $\frac{1}{5}$ of that, and a farthing = $\frac{1}{16}$ of that. Thus—

1*s*. 8*d*.	$\frac{1}{12}$	8794			@ 2*s*. 0¼*d*.
4*d*.	$\frac{1}{5}$	732	16	8	
¼*d*.	$\frac{1}{16}$	146	11	4	
		9	3	2½	
		£888	11	2½	

If the price be less than a shilling the best way is to take parts of a shilling so as to find the price in shillings, and afterwards reduce to pounds. Thus—

6*d*.	½	10374		@ 9¾*d*.
3*d*.	½	5187		
¾*d*.	¼	2593	6	
		648	4½	
	2,0)	842,8	10½	
		£421 8	10½	

If the price be an even number of shillings the simplest way is to work in florins, and then double the odd florins to get the odd silver in shillings. If the number of shillings be odd, the odd shilling should be taken separately. Thus—

		2719	@ 17*s.*
		8	
@	16*s.*	2175	4
@	1*s.*	135	19
		£2311	3

The first line of the sum is got as follows. 16*s.* is 8 florins or $\frac{8}{10}$ of 1*l.* If, therefore, I multiply by 8 and mark off one figure, I shall have the cost in pounds and florins. But as I want to write down the cost in pounds and shillings and not in pounds and florins, I double the figure which I mark off.

There are certain sums which are better done by subtraction than by addition. Take for instance 6759 @ 18*s.* 4*d.*: I observe that 18*s.* 4*d.* is 1*s.* 8*d.* short of 1*l.*, I therefore subtract $\frac{1}{12}$, thus—

$\frac{1}{12}$	6759	
	563	5
	£6195	15

We must otherwise have taken it either as 10*s.*, 5*s.*, and 3*s.* 4*d.*, or as 10*s.*, 6*s.* 8*d.*, and 1*s.* 8*d.*

Where the price is several pounds there is, of course, no escaping the multiplication; but even here it is sometimes possible to find short cuts. Take for instance 1,193 at 7*l.* 17*s.* 6*d.* I observe that 14*s.* is $\frac{1}{10}$ of 7*l.*, and that 3*s.* 6*d.* is $\frac{1}{4}$ of 14*s.* The sum stands thus—

		1193		
		7		
14*s.*	$\frac{1}{10}$	8351		
3*s.* 6*d.*	$\frac{1}{4}$	835	2	
		208	15	
		£9394	17	6

Practice may also be applied to weights or measures, as well as to money. It will in these cases be necessary to use different divisors, but the principle is the same. Let it be required, for instance, to find the gross weight of 179 cast-iron pipes, each weighing 2 cwt. 3 qrs. 7 lbs. The work is as follows:

2 cwt. 2 qrs.		179		
1 qr. 7 lbs.		22	7	2
		2 15	3	21
	Tons	25 2	3	21

Cases occasionally present themselves of double practice. Take, for example, 184 tons 17 cwt. 3 qrs. 14 lbs. of copper, @ £87 17*s.* 11*d.* per ton.

In reality we make three sums of this. We take

184 tons @ £87.
184 tons @ 17*s.* 11*d.*
17 cwt. 3 qrs. 14 lbs. @ £87 17*s.* 11*d.* per ton.

184
87
1288
1472
£16008

10*s.*	½	184 tons @ 17*s.* 11*d.*		
5*s.*	½	92		
2*s.* 6*d.*	½	46		
5*d.*	$\frac{1}{6}$	23		
		3	16	8
		£164	16	8

10 cwt.	½	87	17	11
5 cwt.	½	43	18	11½
2½ cwt.	½	21	19	5¾
1 qr.	$\frac{1}{10}$	10	19	9
14 lbs.	½	1	1	11¾
			10	11¾
		78	11	1¾
		164	16	8
		16008		
		£16251	7	9¾

There is one general rule in practice, and that is to take the quickest way of getting the work done. It is quite possible to spend more time in looking for the shortest way, than is needed to take the work straightforward.

Examples in Practice.

1. 10,000 @ $8\frac{1}{2}d.$
2. 7895 @ $10\frac{1}{4}d.$
3. A million @ $11\frac{3}{4}d.$
4. 31416 @ 2s. 11d.
5. 21828 @ 14s. 9d.
6. 1745 @ 19s. 11d.
7. 57712 @ 16s.
8. 23424 @ 17s.
9. 70103 @ 18s. 6d.
10. 37712 @ 16s. $10\frac{1}{2}d.$
11. A mile @ 3s. 11d. a yard.
12. A ton @ 8s. $7\frac{1}{2}d.$ a lb.
13. A lb. troy @ $1\frac{3}{4}d.$ a grain.
14. An acre at a penny farthing a foot.
15. Two miles 1331 yards @ 1s. 10d. a pole.
16. The weight of a cubic yard @ 1 lb. 13 oz. to the inch.
17. The weight in tons of 257,354 cubic feet of sea-water @ 64 lbs.
18. The rateable value of a parish is £37,350. What will a rate of 1s. $5\frac{1}{2}d.$ in the pound bring in?
19. What will an income tax of 5d. in the pound give on an income of £27,000,000?
20. 3754 oz. of gold @ £3 17s. $10\frac{1}{2}d.$
21. The weight of a sheet of lead measuring $35 \times 10\frac{1}{2}$ feet @ 6 lbs. 11 oz. to the foot, and its cost @ $3\frac{1}{2}d.$ per lb.
22. The weight of 3 miles of iron pipes 189 lbs. per foot run, and the cost @ 8s. 10d. per cwt.
23. What is the difference of cost between 1400 tons of cast iron @ £5 11s. 6d., and 850 tons of wrought iron @ £9 8s. 6d.?

24. What is the difference of weight, in tons, &c., between 100 fathoms of chain cable weighing 76 lbs. to the foot, and 300 fathoms wire rope at 18 lbs.? What will be the difference of cost, the chain cable being 15*s.* 6*d.* per cwt. and the wire rope £1 3*s.* 6*d.*?

25. 512 tons 16 cwt. 3 qrs. 21 lbs. @ £9 3*s.* 6*d.* per ton.

26. Breaking metal for 176 miles 3 fur. 19 chains of road @ 3½*d.* per yard run.

27. 113 acres 3 roods 27 poles @ 3*s.* 10*d.* per sq. foot.

28. Make out the following bills and invoices

34 lbs. 14 oz. beef @ 8½*d.*
14 lbs. 10 oz. leg mutton @ 9½*d.*
7 lbs. 6 oz. chops @ 11½*d.*
15 lbs. 7 oz. lamb @ 1*s.* 2½*d.*
8 lbs. 9 oz. suet @ 8*d.*

29. Bricklayer:

120 rods 95 feet reduced brickwork in mortar @ £9 16*s.* per rod of 272 feet.
97 yards brick nogging @ 1*s.* 5½*d.*
660 feet drains in cement @ 8½*d.*
127 yards super facings with picked grey stocks @ ¾*d.* per foot super.

30. 115 yards Brussels carpet @ 5*s.* 10*d.*
95 yards Dutch stair @ 2*s.* 7*d.*
84 yards Kidderminster @ 3*s.* 7*d.*
72 yards drugget @ 2*s.* 8*d.*
Making @ 3*d.* a yard.
Stair-rods 10 doz. @ 5*s.* 6*d.*
Eyes 20 doz. @ 9*d.*

31. 750 tons iron plates @ £8 15*s.* 6*d.*
215 tons angle iron @ £11 14*s.*
20 tons bulb iron @ £10 10*s.*
17 tons T iron @ £15 10*s.*
96 tons steel @ £18 7*s.* 6*d.*
30 tons rivets @ £22 1*s.*

32. Valuation of an estate:

	£	s.	d.
House and stabling	750	0	0
Farm buildings and cottages . . .	950	0	0
Garden and orchard 7 a. 3 r. 17 p. @ £275 per acre			
Arable 517 a. 2 r. 25 p. @ £125 . .			
Meadow 37 a. 1 r. 18 p. @ £180 .			
Coarse Pasture 751 a. 3 r. 4 p. @ £65 .			
Woodland, old, 25 a. 2 r. 19 p. @ £90 .			
,, new plantations 85 a. 3 r. 17 p. @ £75			
Bog 27 a. 2 r. 27 p. @ £45 . . .			
Foreshore to low water mark 210 a. @ £2 12*s.* 6*d.*			

33. 12 tons 13 cwt. 3 qrs. 4 lbs. copper @ 95*s.* 6*d.* per cwt.
7 tons 17 cwt. 2 qrs. spelter @ 37*s.* 9*d.* per cwt.
2 tons 9 cwt. 1 qr. 22 lbs. tin @ 75*s.* 4*d.* per cwt.
87 tons coal @ 9*s.* 10*d.*
Loss of metal in casting $\frac{1}{20}$.
Labour equal to 1 man for 27 days @ 4*s.* 6*d.*

What is the total cost, and what is the net cost per ton of the finished work?

CHAPTER XI.

FACTORS, MULTIPLES, AND DIVISORS.

Definition.—A measure of a number is any number which will divide it without a remainder. Thus, 4 measures 28 because it divides it with a quotient 7 and no remainder. 2, 7, and 14 also measure 28. A measure of a number is also sometimes called a factor.

Every number is measured by itself and by unity. There

are some numbers which cannot be measured by any other number than themselves and unity, such as 5, 13, 17, 19, 61, &c. These are called prime numbers, or simply primes.

All other numbers which can be split up, such as $21=3\times 7$, $10=2\times 5$, $1001=7\times 11\times 13$, $81=3\times 3\times 3\times 3$, are called composite numbers. The parts into which they can be split up are, as already stated, factors or measures. Thus 1, 2, 3, 4, 6, 8, 12, and 24, each of them measure 24.

On the other hand, 24 is a multiple of each of these numbers except the last. We do not say that a number is a multiple of itself.

All the numbers which are divisible by 2 are called even numbers, and all the others are called odd numbers. Evidently no even number is prime except 2 itself. It is also evident that any multiple of a number is measured by every factor of that number. For instance, since 35 is divisible by 7, so is 11×35. For if we write the first number as 5×7 our introducing another factor as $5\times 7\times 11$ still leaves 7 a factor.

It follows that all numbers ending with 0, 2, 4, 6, 8, are even. Take, for example, 56; this may be written as $5\times 10+6$, and since 10 and 6 are each divisible by 2, so is the whole number. The same reasoning may be applied to any number ending in 0, 2, 4, 6, or 8, and the principle may therefore be considered as established. In the same way, it is easy to see that any number ending in 5 or zero is a multiple of 5; moreover, no other numbers are. We are thus enabled to tell at sight whether a number is or is not divisible by 2 or 5. We are enabled to do this because our notation is founded on the number $10=2\times 5$.

In what is called the proof of nines, we remarked that any number whatever divided by 9 left the same remainder as if we added together all the figures of which it consisted, and divided that by 9. Hence, if on casting out the nines we get no remainder, the number is measured by 9; and in this case, or if we get 3 or 6 as a remainder, the number is measured by 3. For then the number may be written in

one of the following ways: a multiple of 9, 3 + a multiple of 9, or 6 + a multiple of 9; and since 9 = 3 × 3 this proves the statement. If the number is measured by 6, it must also be an even number. We can thus ascertain, almost at sight, whether a number is measured by 3, 6, or 9, and absolutely at sight whether it is measured by 2 or 5. For other numbers we can do nothing simpler than to try them by actual division. For very large numbers, however, we may ascertain whether they are divisible by 7, 11, or 13, by dividing at once by their product—namely, 1001. This division is very easy to perform, and we can then try whether the remainder from it (if any) is divisible by either separately.

Common Multiple and Common Measure.

If one number measures two or more other numbers it is said to be a common measure of those numbers. Thus, 2, 3 and 6 are each of them common measures of 42 and 72, 3 is a common measure of 42, 72, and 39.

Numbers are said to be prime to one another when they have no common measure; for example, 35 and 22 are prime to one another, although neither is a prime number.

If a number be measured by two or more other numbers it is said to be a common multiple of those numbers. Thus, 70 and 140 are each of them common multiples of 14 and 5. So 420 is a common multiple of 2, 3, 4, 5, 6, 7, 10, 12, 14, 15, 20, 21, 28, 30, 35, 42, 60, 70, 84, 105, 140, and 210.

Whenever we are able to split two numbers up into their prime factors, that is into factors which we cannot split up any further, it is very easy to find as many of them measures and multiples as we desire, and we can pick out, what is of some importance, the greatest measure of the two and the least multiple of the two. It is evident that these are the most useful among all the common measures and multiples. Take for instance the case of 42 and 72; 42 is 2 × 3 × 7; 72 is 2 × 2 × 2 × 3 × 3. In the former, 2, 3, and 7 each occur once. In the latter, 2 occurs 3 times, 3 occurs twice, and 7

not at all. Now, as 7 occurs only in one of them it cannot enter into their common measure, but 2 and 3 measure them both. Again, 2 only measures 42 once, that is to say, it does not measure the quotient 21. Therefore, in the common measure the 2×3 can occur only once. Hence, their product, 6, is not only a common measure but is also the greatest common measure of 42 and 72.

Let us next consider how to form the least common multiple. Since 2 occurs three times in one of them and once in the other, it must occur three times in any common multiple and it need not occur more than three times in the least common multiple. Similarly the 3, which occurs twice in one of them, must occur twice in any common multiple, and only twice in the least common multiple. The 7 which occurs singly must appear in any common multiple, and can only appear once in the least. If any other prime number than these enters into the multiple, that will not be the least. Hence, finally, we arrive at the conclusion that the least common multiple not only contains, but consists of 2 taken three times, 3 taken twice, and 7; or, in other words, that it is $2 \times 2 \times 2 \times 3 \times 3 \times 7 = 504$.

If we compare this least common multiple of two numbers with their product, we shall find that the least common multiple is the product divided by the greatest common measure. For instance, $504 = \frac{3024}{6} = \frac{42 \times 72}{6}$.[1]

It is not always possible to split up numbers at sight. For instance, it would take a great many trials to find out that 1763 is 41×43, or that 2911 is 41×71. It has therefore been thought worth while to find some rule for obtaining the G. C. M. of two numbers when we do not know how to split them into factors. The rule depends on the following principles: *Firstly,* that if two quantities have a common measure, it also measures the remainder, if any, which arises

[1] Let $k \times x$ and $k \times y$ be two numbers of which k is the G. C. M. Then it is evident that $k \times x \times y$ is their L. C. M. Their product is $k \times k \times x \times y$, and if we divide this by k we get the L. C. M.

from dividing the greater by the lesser. For if we take the greatest common measure of two numbers as a unit, and express each of the numbers in that unit, the remainder will also be expressed in terms, not fractional, of that unit. For example, consider any two numbers of inches, such that 12 is a common measure of them; then we can express each of them as a whole number of feet; and if we divide one by the other the remainder must also be a whole number of feet. That is to say, if we express it in inches it must be measured by 12. *Secondly*, since this is true of any common measure it is also true of the greatest.

The rule for finding the G. C. M. of two quantities is as follows. Divide the greater by the less until we come to a remainder smaller still. This remainder either is the G. C. M. or contains it as a factor. If it is the G. C. M. it will divide the lesser number without a remainder. If not, since the first remainder and the less number each contain the G. C. M., it must also be contained in the second remainder. If, then, the second remainder measures the first it is the G. C. M., if not we must continue the process: the last remainder found will be the G. C. M. If the last remainder be unity, the numbers are said, somewhat inaccurately, to have no common measure; strictly, none but unity.

```
1763 ) 2911 ( 1
       1763
       ----
       1148 ) 1763 ( 1
              1148
              ----
               615 ) 1148 ( 1
                      615
                      ---
                      533 ) 615 ( 1
                            533
                            ---
                             82 ) 533 ( 6
                                  492
                                  ---
                                   41 ) 82 ( 2
                                        82
```

In this example 41, being the last remainder, is the G. C. M.

Next let it be required to find the G. C. M. of 68590142 and 85044059. The work at full length is:

```
68590142)85044059(1
         68590142
         16453917) 68590142(4
                   65815668
                    2774474) 16453917(5
45935)73496(1                13872370
      45935                   2581547) 2774474(1
   27561) 45935(1                      2581547
          27561                         192927) 2581547(13
          11374) 27561(1                         192927
                 18374                            652277
                  9187) 18374(2                   578781
                        18374                      73496) 192927(2
                                                          146992
                                                           45935
```

The work may very often be abbreviated by the remark that if one of the numbers is observed to contain a factor which is prime to the other, that factor cannot appear in the common measure, and may therefore be divided out at once. Thus, the previous sum may be worked as follows. One number being odd and the other even, the G. C. M. must be odd, and we may divide out the 2; we then get:

```
34295071 ) 85044059 ( 2
           68590142
           16453917 ) 34295071 ( 2
                      32907834
                       1387237 ) 16453197 ( 11
                                 1387237
                                  2581547
                                  1387237
                                  1194310
```

I observe that this remainder is divisible by 10, while the

previous one is not divisible by either 2 or 5. Dividing out the 10 I go on again:

```
119431 ) 1387237 ( 11
         119431
         ------
          192927
          119431
          ------
           73496
```

This remainder again is divisible by 8, while the last remainder is odd. Dividing out the 8, I get:

```
9187 ) 119431 ( 13
       9187
       -----
       27561
       27561
```

Wherefore 9187 is the G. C. M.

It is always worth while to verify by actual division whether the G. C. M. which has been found really does measure the given numbers. For instance, in this case, $\frac{68590142}{9187} = 7466$, and $\frac{85044059}{9187} = 9257$.

It has been stated that the L. C. M. of two numbers is their product divided by their G. C. M. This involves more work than is necessary; the practical rule is to divide one of the numbers by the G. C. M. and multiply the other by the quotient. It is evident that this comes to the same thing. Suppose, for instance, we wish to find the L. C. M. of 68590142 and 85044059, of which we have just found the G. C. M. to be 9187. The latter of them may be written as 9187×9257, and their product may be written as $9187 \times 9257 \times 68590142$. We may now divide by the G. C. M. by simply striking out the factor 9187. What this leaves us is the quotient 9257×68590142. The L. C. M. is therefore—

$$9257 \times 68590142 = 634938944494.$$

To find the greatest common measure of three or more

quantities, take any two of them and find their G. C. M. Then find the G. C. M. of this, and the third; then the G. C. M. of this, and the fourth. The reason for the rule is obvious. As an example, take 42, 56, and 63; the G. C. M. of 42 and 56 is 14; the G. C. M. of 14 and 63 is 7; therefore 7 is the G. C. M. of the three. We can verify this by splitting them into factors, thus:

$$42 = 3 \times 2 \times 7$$
$$56 = 2 \times 2 \times 2 \times 7$$
$$63 = 3 \times 3 \times 7$$

and it is easily seen that 7 is the only common measure besides unity. The rule for the L. C. M. of three or more numbers is nearly similar. Find the L. C. M. of any two, then of that L. C. M. and the third, then of that L. C. M. and the fourth, and so on.

Thus, in the case of 42, 56, and 63, the G. C. M. of 42 and 56 is 14; therefore the L. C. M. of 42 and 56 is

$$\frac{42}{14} \times 56 = 3 \times 56 = 168.$$

The G. C. M. of 168 and 63 is 21; therefore the L. C. M. is

$$\frac{168}{21} \times 63, \text{ or } 8 \times 63 = 504.$$

This is easily verified by their factors, which have been given above.

A mistake sometimes made by beginners is to extend to the L. C. M. of three or more numbers the principle which holds for two, viz.

$$\text{L. C. M.} = \frac{\text{product}}{\text{G.C.M.}}$$

but this is not true for more than two numbers. For example, in the case above given of 42, 56, and 63, of which the G. C. M. is 7, this rule would give us $6 \times 56 \times 63$; whereas we know that the *least* common multiple is only 8×63.

CHAPTER XII.

FRACTIONS.—PART II.

WE are now in the condition to complete our chapter on the Reduction of Fractions. We begin with the simplification of fractions. A fraction is said to be reduced to its lowest terms when its numerator and denominator contain no common factor except unity. Thus, $\frac{3}{4}$ and $\frac{12}{23}$ are already in their lowest terms, but $\frac{42}{63}$ is not so, because the factor 21 can be divided out. The rule for effecting the reduction is to find the G. C. M. of the numerator and the denominator, and then divide it out. For the G. C. M. includes all the factors common to both of them. Thus, in the case of $\frac{68590142}{85044059}$ we have seen that 9187 is the G. C. M. of the numerator and denominator. Dividing out this factor the fraction becomes $\frac{7466}{9257}$. It is now in its lowest terms.

To reduce fractions to a common denominator, it is sufficient to find *any* common multiple of the denominators; but as it is an advantage to keep numbers as small as possible, it is evidently better in general to use their L. C. M. The rule, then, for reducing fractions to their lowest common denominator is to find the L. C. M. of the denominators and to multiply the numerator of each fraction by the quotient of the L. C. M. divided by its own denominator. Thus, in the case of $\frac{11}{42}$, $\frac{9}{56}$, $\frac{8}{63}$, the L. C. M. of the denominators is 504. This is 12×42; hence $\frac{11}{42}$ is to be written as $\frac{132}{504}$. Again, $504 = 9 \times 56$; therefore $\frac{9}{56}$ must be written as $\frac{81}{504}$. Further, 504 is 8×63; therefore $\frac{8}{63}$ must be written $\frac{64}{504}$. If, now, we want to perform the addition $\frac{11}{42} + \frac{9}{56} + \frac{8}{63}$, we can do so with the least possible expenditure of figures by adding the equivalent fractions, $\frac{132}{504} + \frac{81}{504} + \frac{64}{504} = \frac{277}{504}$. It happens that this fraction $\frac{277}{504}$ is already in its lowest terms, but this is not necessary in such cases. For instance, using

the lowest common denominator, $\frac{1}{2}+\frac{1}{6}=\frac{3}{6}+\frac{1}{6}=\frac{4}{6}$, which is not in its lowest terms.

It is not necessary to say much about subtraction, or mixed addition and subtraction. When we have once reduced the fractions to a common denominator we have simply to perform these operations upon the numerators. Thus, $\frac{11}{42}+\frac{9}{56}-\frac{8}{63}$ is $\frac{132}{504}+\frac{81}{504}-\frac{64}{504}=\frac{149}{504}$.

The greatest common measure of fractional quantities, and their least common multiple, are vague expressions; because if fractions are to be admitted, all numbers are divisors and multiples of every other number. Nevertheless, there may be a reasonable meaning in certain cases. For instance, $\frac{2}{15}$ and $\frac{8}{35}$ might reasonably be said to have $\frac{2}{5}$ for a greatest common measure; and $\frac{8}{105}$ might be said to be in some sense their least common multiple. But this is very much a question of terms, or of the intention with which the reduction or comparison is instituted, rather than a matter for a fixed rule.

1. Write down all the divisors of

(1) 72	(2) 2240	(3) 5760
(4) 5670	(5) 2646	(6) 77077

2. Write down all the common divisors of

(1) 5760 and 5670	(4) 338, 416 and 8008
(2) 720 and 3096	(5) 847, 539 and 616
(3) 98, 2240 and 5670	(6) 216, 3096 and 2908

3. What factors must be used to multiply each of the numbers in the first of the following columns to produce the one opposite to it in the second column?

32	1024
128	1728
625	256
692	69892
36801	261
114681	379476

4. Find the greatest common measure of

(1) 32 and 56
(2) 120 and 72
(3) 567 and 576
(4) 1001 and 1331
(5) 729 and 999
(6) 1296 and 4096
(7) 3737 and 5476
(8) 4343 and 121

5. Find the greatest common measure of

(1) 2521777 and 36268013
(2) 48249049 and 48530761
(3) 1761151 and 1747963
(4) 860149 and 2006153
(5) 2442431 and 1562009
(6) 1718446 and 4439618
(7) 726221432301 and 432301726221
(8) 318732033187 and 320331873203

6. Reduce to their lowest terms the fractions got by dividing one by the other the numbers given in the Examples of sets 4 and 5. For instance, reduce $\frac{32}{56}$, $\frac{120}{72}$, and $\frac{2521777}{36268013}$, and so forth to their lowest terms.

7. Find the least common multiple of the numbers given in set 4, and in the first two examples in set 5.

8. Find the greatest common measure and least common multiple of the following sets of numbers:

(1) 75, 125, 1000
(2) 54, 96, 64
(3) 48, 72, 162
(4) 30, 60, 90, 150, 180, 300, 450, 900
(5) 343, 5929, 7007, 8281, 16807
(6) 13792381, 32080621, 33452239, 5806592401

9. Obtain the results of the following additions and subtractions in their lowest terms:

(1) $\frac{1}{25928}+\frac{1}{4344}$
(2) $\frac{1}{25928}-\frac{1}{4344}+\frac{1}{76801}$
(3) $\frac{47}{1363}+\frac{43}{1677}+\frac{41}{2091}$
(4) $\frac{1}{14701}-\frac{1}{17892}-\frac{1}{19639}+\frac{1}{20603}$

(5) $1+\frac{1}{24}-\frac{17}{5760}+\frac{367}{1890}-\frac{27859}{68700}$
$+\frac{1295803}{18711100}-\frac{15183676231}{15230900700}$

(6) $\frac{1}{3}-\frac{22}{135}+\frac{1528}{14175}-\frac{119856}{1488375}$

(7) $1+\frac{1}{6}+\frac{1}{24}+\frac{61}{5040}+\frac{277}{72576}$

(8) $1-\frac{1}{6}+\frac{1}{24}-\frac{61}{5040}+\frac{277}{72576}$

10. Reduce to their lowest terms the following fractional expressions:

(1) $\dfrac{19l.\ 16s.\ 7\frac{3}{4}d.}{20l.\ 16s.\ 8\frac{3}{4}d.}$

(2) $\dfrac{217 \text{ miles } 5 f.\ 18 p.\ 2 yd.\ 2 ft.\ 1 in.}{506 \text{ miles } 2 f.\ 23 p.\ 1 yd.\ 2 ft.\ 7 in.}$

(3) $\dfrac{767 \text{ acres } 9 \text{ chains } 279 yds.\ 4 ft.}{208 sq.m.\ 181 \text{ acres } 0 \text{ chains } 93 yds.\ 4 ft.}$

(4) $\dfrac{70 \text{ tons } 7 cwt.\ 0 qrs.\ 27 lbs.\ 1 oz.}{1011 \text{ tons } 18 cwt.\ 3 qrs.\ 10 lbs.\ 13 oz.}$

11. Find the values of

(1) $\dfrac{\frac{17}{10}+\frac{7}{11}}{\frac{17}{10}-\frac{7}{11}}-\dfrac{9+\frac{14}{17}}{2+\frac{2}{5}}$

(2) $\dfrac{\frac{1}{3}+\frac{1}{5}+\frac{1}{7}}{\frac{1}{2\frac{1}{2}}+\frac{1}{3\frac{1}{2}}+\frac{1}{4\frac{1}{2}}}\times\frac{13}{71}$ of $7\frac{1}{3}$

(3) $1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{5+\cfrac{1}{5}}}}}$

(4) $1+\cfrac{2}{3+\cfrac{4}{5+\cfrac{6}{7+\cfrac{8}{9}}}}$

(5) $1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2}}}}}$

(6) $\dfrac{\frac{1}{2\frac{1}{2}}+\frac{1}{4\frac{1}{4}}+\frac{1}{6\frac{1}{6}}}{\frac{1}{3\frac{1}{3}}+\frac{1}{4\frac{1}{4}}+\frac{1}{5\frac{1}{5}}}$

12.

(1) What fraction of 17*s.* is $\frac{3}{4}$ of 2*s.* 10*d.*?

(2) What fraction of a pole is $\frac{3}{11}$ of a yard?

(3) What fraction of a yard is $\frac{4}{11}$ of a pole?

(4) What fraction of a lb. troy is $\frac{9}{14}$ of a lb. avoirdupois?

(5) What fraction of a scruple is a pennyweight?

(6) What fraction of a square yard is $\frac{7}{110}$ of a pole?

(7) What fraction of a million inches is $\frac{15}{121}$ of a mile?

(8) What fraction of 974153 is 988027?

(9) What fraction of 25 tons 6 cwt. 3 qrs. 10 lbs. 11 oz. is 23 tons 11 cwt. 1 qr. 1 lb. 1 oz.?

(10) What fraction of 439 miles 7 furlongs and 211 yards is 445 miles 21 yards?

(11) Express $1\frac{2}{59}$ tons + $\frac{47}{97}$ cwt. + $\frac{5224}{8633}$ lbs. in cwt.

(12) Express $\frac{3}{40}$ ton + $\frac{5}{14}$ cwt. + 11 [illegible] lbs. in tons.

13.

(1) At what times are the hands of a clock exactly one over the other?

(2) A reservoir has two sluices, one of which alone would drain it in 7 hours, and the other in 13. How soon would it be emptied if both were opened together?

CHAPTER XIII.

DECIMAL FRACTIONS.

In any system of Decimal notation it is a mere matter of convenience what we shall take for our unit. But when we have once named the unit, and marked the unit column of any series of figures, the tens, hundreds, thousands, and so forth, fall naturally into their places; the tens being in the first place to the left of the unit, the hundreds the second, and so on.

Now consider in French money 16 francs 75 centimes; which by Reduction we know is 1675 centimes. If we con-

sider the centime as the unit of account the unit figure is 5 If we take a franc as our unit the unit figure is 6; and in that case the figures to the right of the unit figure, viz. 75, represent some fraction of a franc, the figure 7 standing for seventy centimes or $\frac{7}{10}$ of a franc, and the 5 representing five centimes or $\frac{5}{100}$ of a franc.

In this method of notation it is clear that we shall want a special mark to indicate the unit figure, in order to enable us to distinguish between the 1675, which stands for sixteen francs and a fraction, and the 1675 which stands for one thousand six hundred and seventy-five francs. The usual way of doing this is to put a dot to the right of the unit figure: thus, 16·75. This dot is called the decimal point. Its sole purpose is to mark the unit figure.[1]

There is no mystery at all about decimals. They are simply the extension of the ordinary system of integral notation, in which the value of a figure depends upon its place, to the division of a unit, instead of restricting it to the multiplication of units.

In reading any series of figures from left to right, each figure has only $\frac{1}{10}$ of the value which it would have if it were one place to the left. Thus, in 777, the second 7, which means *seventy*, is $\frac{1}{10}$ of the *seven hundred*, which is indicated by the first 7. . So again the third 7, which is simply *seven units*, is $\frac{1}{10}$ of the *seventy* which is represented by the second 7. What is there to prevent us, if we want to mark smaller intervals than whole units, from carrying on this principle and using a fourth 7 to represent $\frac{1}{10}$ part of the seven units; or (what is the same thing) $\frac{7}{10}$ of one unit? For this purpose two things are necessary, and two only. First to write the fourth 7 to the right of the third. Secondly, since the unit place is no longer the last figure to the right, to use some special mark for it. This mark is the decimal point.

[1] The decimal point should be put at the top of the line of figures, thus—5·7, because 5 . 7 with the stop at the bottom is used in most works to mean 5 × 7 or 35.

The figures to the left of the decimal point represent units, tens, hundreds, thousands, &c., in the ascending scale, the figures to the right of it represent tenths, hundredths, &c., in the descending scale. Thus:

$$777{\cdot}777 = 700 + 70 + 7 + \tfrac{7}{10} + \tfrac{7}{100} + \tfrac{7}{1000}$$
$$691{\cdot}324 = 600 + 90 + 1 + \tfrac{3}{10} + \tfrac{2}{100} + \tfrac{4}{1000}$$
$$70{\cdot}004 = 70 + \tfrac{4}{1000}$$
$${\cdot}005 = \tfrac{5}{1000}$$

The reason for using decimal fractions in preference to other fractions is that the simple rules of arithmetic apply to them without any change at all. The only precaution needed is not to lose sight of the unit figure. In adding decimal fractions, for instance, we 'carry' just as if we were working with whole numbers, and, in fact, we need not concern ourselves with there being fractions at all, provided we only keep clearly in sight which is the unit figure.

For instance, let us add the four lines given above: both the right-hand and left-hand expressions. Beginning on the right we have $\frac{20}{1000}$, which is simply $\frac{2}{100}$ without any odd thousandths; carrying this to the hundredths column, as we may obviously do, we get, on adding up that column, $\frac{11}{100}$, which is the same as $\frac{1}{10} + \frac{1}{100}$. We have, therefore, an odd $\frac{1}{100}$. Including the $\frac{1}{10}$ the next column adds to $\frac{11}{10}$; the 1 is one unit, and we have therefore 9 for the unit figure of the sum; the rest of the work is simple addition, and we get finally $1539 + \frac{1}{10} + \frac{1}{100}$. Now adding up the left-hand side, and simply following the ordinary rule of keeping units under units, tens under tens, and so forth, we obtain 1539·11.

We do not write ·110, because the absence of the 0 indicates the same thing as its presence, viz. that there are no odd 1000ths. But if there had been more figures to the right—say some odd 10,000ths—we should then have wanted it to keep the place. For example, 7·11 means the same as 7·110. But evidently 7·117 does not mean the same as 7·1107. In the first case, the final 7 means $\frac{7}{10}$ of the figure *one*, which goes before it and which is itself one-tenth of a unit. In the second case, it means $\frac{7}{100}$ of that same $\frac{1}{10}$. We will now state formally the principles of decimal fractions.

Notation.

1. The unit figure is marked by having a stop placed after it (near the top of the figure) called the decimal point.

2. Shifting a figure one place to the left multiplies it by 10, and shifting it one place to the right divides it by 10, as in ordinary notation. But while, in the ordinary notation, as we go to the right we stop at the unit point, in using decimal fractions we do not stop at the unit point, but we carry on the descending scale as far as we please to the right.

&c.	thousands	hundreds	tens	units		tenths	hundredths	thousandths	ten thousandths	hundred thousandths	&c.
	6	7	9	1	·	8	0	7	3	2	

This is read as six thousand seven hundred and ninety-one *decimal* eight nought seven three two. Expressed in vulgar fractions, it may be written indifferently in three ways :

$$6791 + \frac{8}{10} + \frac{0}{100} + \frac{7}{1000} + \frac{3}{10000} + \frac{2}{100000}$$

$$6791\frac{80732}{100000} \qquad \text{or} \qquad \frac{679180732}{100000}$$

It is evident that noughts on the extreme left, or on the extreme right, of any set of figures are only of use where the unit point cannot be properly marked without them. Thus in 0079·3200 the noughts are useless; but in the case of 7900· they are needed to distinguish seven thousand nine hundred from seventy-nine, and conversely in ·0032 they are needed to distinguish $\frac{3}{1000} + \frac{2}{10000}$ from $\frac{3}{10} + \frac{2}{100}$.

Conversion of a Decimal into a Vulgar Fraction.

Take the decimal fraction itself for the numerator of the vulgar fraction, and for the denominator write down as many noughts

as there are decimal places and put down the figure 1 to the left of them. Thus, $\cdot 76=\frac{76}{100}$, $\cdot 762=\frac{762}{1000}$, $\cdot 706=\frac{706}{1000}$, $\cdot 0045=\frac{45}{10000}$, $\cdot 00003045=\frac{3045}{100000000}$.

This is a simple consequence of the decimal notation, of which it is in fact only a statement in another form. When we have converted them into vulgar fractions this way, we have still to reduce these fractions to their lowest terms. For instance, $\cdot 4=\frac{4}{10}=\frac{2}{5}$, $\cdot 5=\frac{5}{10}=\frac{1}{2}$, $\cdot 25=\frac{25}{100}=\frac{1}{4}$, $\cdot 075=\frac{75}{1000}=\frac{3}{40}$, $\cdot 00032=\frac{32}{100000}=\frac{1}{3125}$, $\cdot 3125=\frac{3125}{10000}=\frac{5}{16}$.

Addition and Subtraction of Decimals.

This is the same as for whole numbers, and needs the same precaution to keep the unit's place in the same column, or, what amounts to the same thing, keeping the decimal points one above the other. Only this is no longer the same thing as keeping the right-hand figures in one column, as when we are adding whole numbers.

Examples of addition:

```
1234·6789          66199·3226
  13                   ·301
 170                 54·5
    ·0054               ·00632
    ·5                1000
  87·142                ·07
---------         32745·80008
1505·3263         -----------
                  100000
```

Examples of subtraction of decimals:

```
98765·4321        5·1394
   99·99           ·97658
----------        -------
98665·4421        4·16282

·01               1
·0000999           ·47712
--------          -------
·0099001           ·52288

347·258745        1
247·258746         ·000001
----------        --------
 99·999999         ·999999
```

Multiplication of Decimals.

The same rules apply as to the multiplication of whole numbers. The only difficulty in the case of decimals is to find where the unit point of the product is. Suppose, for instance, we have 12·57 to multiply by 9·4. As far as the figures are concerned the multiplication is simply as follows:

```
 1257
   94
 ----
 5028
11313
-----
118158
```

We have now to fix the unit or decimal point. For this purpose I observe that 12·57 lies between the whole numbers 12 and 13, and that 9·4 lies between the whole numbers 9 and 10. Hence the product $12·57 \times 9·4$ lies between 12×9 and 13×10; that is to say, between 108 and 130. This shows us that the unit figure in the product $12·57 \times 9·4$ is the third figure going from left to right, and we therefore put the decimal point between the third and fourth figure, thus: 118·158.

This method of determining the decimal point is not the most convenient in all cases. A more specific rule is: *To find the number of decimal places in the product, add the number of decimal places in the multiplier and multiplicand;* thus, in the foregoing example there are two decimals in the multiplier and one in the multiplicand, therefore there will be $2+1=3$ in the product.

We may see the reason for this as follows: 1257 is $100 \times 12·57$; 94 is $10 \times 9·4$. Therefore, $1257 \times 94 = 1000 \times 12·57 \times 9·4$; so that 118158 is 1000 times the product that we want to get, and therefore we must mark off three figures as decimals. It is easy to see that the rule is general, because we have done the multiplication as if the multiplier and

multiplicand were whole numbers, and to do this we have shifted forward the decimal point two places in one and one in the other, or three places altogether; and therefore we must shift it back three places in, or cut off three decimals from, the quotient.

Example, ·008 × ·0007.

The multiplication of the figures 8 and 7 gives 56, and from this we have to cut off $3+4=7$ figures of decimals. To do this we must put a string of noughts on the left-hand side of the 56, thus, 00000000056: we can now cut off the 7 figures thus, 0000·0000056, or, neglecting the noughts to the left of the decimal point, our product is ·0000056. This is easily verified, for—

$$\cdot 008=\frac{8}{1000} \text{ and } \cdot 0007=\frac{7}{10000};$$

the product of these is—

$$\frac{56}{10000000} \text{ or } \cdot 0000056.$$

Division of Decimals.

Like multiplication, the division of decimals is done primarily with reference to the figures only, and not to their value as depending on the position of the decimal point. As many noughts may be added to the dividend as may be convenient. The rule for pointing is obtained in the same way as in multiplication; that is to say, consider by how many places you have altered the unit point in the dividend to make it a whole number, and so with the divisor. If, for instance, you have shifted your unit ten places in the dividend and three in the divisor, your quotient will be seven places wrong, and the unit place in that must be shifted seven figures to the left accordingly. Suppose, for instance, we have to divide ·7 by ·176; we proceed as follows:

```
176) 700  ( 39772727
     1720
      1360
       1280
        480
        1280
         480
         1280
           48
```

In the above division we have shifted the 7 one place to make it a whole number, and we have added nine noughts, which is equivalent to ten places for the dividend. We have shifted the divisor three places; hence seven must be marked off from the quotient, which is therefore 3·9772727 with a remainder.

Again, to divide ·03759 by 28·7

```
287 ) 3759 ( 1309756
      889
       2800
        2170
         1610
          1750
            28
```

Here we have shifted five places to make 3759 a whole number instead of a fraction, and have added five noughts in the division. On the other hand, we have shifted the divisor one place. The quotient has therefore nine decimal places, and is ·001309756.

We will next divide 314·159 by ·0000008937.

```
8937 ) 314159 ( 3515
        46049
         13640
          47230
           2345
```

Let us stop at this point and consider our result. We have shifted the dividend three places and added two noughts, This is equivalent to a shift of five places. But we have shifted the divisor ten places. We shall therefore have to shift the quotient five places to the left, making it

351500000.

This result is of course inaccurate, but it is sufficient for many practical purposes. To make it complete we must carry on the division: but the *placing* of the figures is settled once for all, for continuing the division will only replace the inaccurate noughts by the actual figures which ought to replace them. The quotient, correct to two decimal places, is

351526239·23.

Conversion of Vulgar Fractions into Decimals.

This is only a particular case of division. For instance, to convert $\frac{17}{8}$ to a decimal, we have simply to add as many noughts as may be required to the numerator and then perform the division:

```
8 ) 17000
    2·125
```

Circulating Decimals.

If the figures in a divisor will not divide the figures in a dividend, the division in general will not come to an end, however many noughts we add to the dividend, but there will always be a remainder unless, indeed, the divisor should

be wholly made up of *twos* and *fives*, either alone or multiplied together, in which case the division will terminate. In the case in which it does not terminate, the quotient figures after a certain time will repeat themselves, coming over and over, thus:

$$3)\,1$$
$$\cdot 33333$$
$$11)\,1$$
$$0\cdot 90,90,90,90$$
$$7)\,1$$
$$0\cdot 142857,142857,1$$

In the above cases all the figures are repeated over and over again, but in others the repeating or *circulating* part is preceded by some decimals which do not repeat. These are called *mixed* circulates. Take, for example, 13÷352; I remark that the denominator is 4 × 8 × 11. Working by short division we have

$$4)\,13$$
$$8)\,3\cdot 25$$
$$11)\,\cdot 40625$$
$$\cdot 0369318181 8$$

It is necessary to notice circulating decimals, because they occur very frequently; but it is not necessary in a practical work to go very far into their theory. It is useful, however, to show how they can be reduced to vulgar fractions. I will begin with a particular class of circulating decimal which arises out of the division of 1 by 9, by 99, by 999, and so on. It is easily seen that $\frac{1}{9}=\cdot 1111$, and so on; $\frac{1}{99}=\cdot 010101$, &c., $\frac{1}{999}=\cdot 001001001$, &c., $\frac{1}{9999}=\cdot 00010001$, &c. The law of these quotients is seen without any difficulty. In the last case, for instance, after adding four noughts to the 1 we have quotient 1 and remainder 1. We get no more quotient figures until we have added four more noughts, and then we

get again quotient 1 and remainder 1 as before. It is evident that these stages repeat themselves exactly at every four figures, as far as we like to carry the division. We can see also that increasing the number of nines in the divisor merely increases the number of noughts between the *ones* in the quotient. Now consider an example like the following: We want to find the vulgar fraction corresponding to ·725725725, &c. This is obviously the product of 725 into ·001001001, &c. But this last we have seen is $\frac{1}{999}$. Therefore the fraction that we want is $\frac{725}{999}$.

We may easily show in the same way that the value of any circulating decimal which begins to circulate from the decimal point, is found by writing down the circulating period as numerator, and, for the denominator, as many nines as there are figures in the period. For instance,

$$\cdot 272727 \text{ is } \frac{27}{99} \text{ or } \frac{3}{11}.$$

A mixed circulating decimal like ·0369318818... must be dealt with by dividing it into two parts, one of which circulates and the other does not; thus

$$\cdot 0369318 18 = \cdot 03693 + \cdot 00000 1818 \ldots$$

$$= \cdot 03693 + \frac{\cdot 1818 \ldots}{100000}$$

$$\frac{\cdot 3693}{100000} + \frac{1}{100000} \times \frac{18}{99}$$

$$\frac{40625}{1100000} = \frac{65}{1760} = \frac{13}{352}$$

The portion which repeats itself is called the period of the fraction. It is usual to write it only once, with a dot over the first and last figure of the period. Thus

$\frac{1}{11} = \cdot 0909 \ldots$	is written as $\cdot\dot{0}\dot{9}$
$\frac{1}{111} = \cdot 009009 \ldots$	is written as $\cdot\dot{0}0\dot{9}$
$\frac{1}{3} = \cdot 3333 \ldots$	is written as $\cdot\dot{3}$
$\frac{1}{24} = \cdot 041333 \ldots$	is written as $\cdot 041\dot{3}$
$\frac{13}{352} = \cdot 036931818.$	is written as $\cdot 03693\dot{1}\dot{8}$

The number of figures in the period is always less than the divisor—that is, than the divisor itself, not the number of figures in it. Circulating decimals possess very many curious properties; but these are generally more curious than useful.

It is to be remarked that ·9999 . . ., indefinitely continued, means unity. Evidently it only differs from unity by a decimal unit with an indefinite number of noughts between it and the decimal point; that is to say, if we only take nines enough, we can make it differ from unity as little as we please.

Discarding Decimals.

It sometimes happens that we desire to retain only a portion of the decimals which we have obtained by some particular process which we have used.[1] It will not do in all cases simply to cut off those that we do not want. In any case we deliberately incur an inaccuracy, which from its smallness we think we can afford to neglect. We must take care, therefore, that in making this error we do it in such a way as to make it the least possible, and also of such a character that it shall not have a tendency to accumulate. What I mean will easily be seen by taking the following addition sum:

$$\begin{array}{r} \cdot 41374 \\ \cdot 82977 \\ \cdot 93489 \\ \cdot 69422 \\ \cdot 90738 \\ \hline 3\cdot 78000 \end{array}$$

[1] It is bad in principle to use more figures than are actually needed in any kind of work; for every additional figure, besides giving needless labour, introduces an additional chance of making a blunder. Students not unfrequently work with ten or fifteen decimals, where the second decimal place is scarcely reliable, simply because they do not know how to get rid of the useless figures. This is not good work. The utmost that should be done is to keep a sufficient margin not to add an error of arithmetic to an uncertainty of measurement.

Now suppose that for some reason we wish to retain four figures of decimals only. Let us try the effect of simply cutting off the last figure of each line:

$$\begin{array}{r} \cdot 4137 \\ \cdot 8297 \\ \cdot 9348 \\ \cdot 6942 \\ \underline{\cdot 9073} \\ 3\cdot 7797 \end{array}$$

Supposing the first to be accurate, the error of the second is ·0003, and we have therefore got our fourth figure of decimals, which we intended to be good, wrong by three units.

There is a much better way of doing it. I observe that 27 is much nearer to 30 than it is to 20, and therefore, if we are to cut off the 7, we shall make a less error if we write 3 than if we write 2 for the previous figure; and this will be the case wherever the figure cut off exceeds 5. In this way there is moreover the advantage that the errors will be some one way, some the other, and they will therefore have a chance of cancelling one another, which they cannot do in the other case, in which they are all one way. Our rule, therefore, should be as follows:—Wherever the figure cut off is less than 5, simply discard it; wherever this figure is 5 or more cut it off, but increase by unity the last figure retained. Thus the example last given should stand:

$$\begin{array}{r} \cdot 4137 \\ \cdot 8298 \\ \cdot 9349 \\ \cdot 6942 \\ \cdot 9074 \\ 3\cdot 7800 \end{array}$$

This comes exactly to the same thing as if we had worked with five figures. It is of course not often that so absolute a coincidence is obtained, but still the accuracy retained by

this simple precaution is very remarkable. The example just given was taken at hazard.

When the last figure happens to be exactly 5, the same error is committed whether we increase the previous figure or not. The reason for putting 5 to the figure above is that if two or more figures are cut off, that is right unless it happens to be followed by noughts: for instance, if we cut three figures off ·0786509, it is a little more correct to make it ·0787 than ·0786. This implies that the figures cut off are really haphazard. Where they are not so it is possible that we may make just as great an error as if we simply discard them. This will be the case if we apply the rule to the halfpence column of a baker's bill.

A similar remark applies to money, or weights and measures. If we propose to *neglect shillings*, we must take 17*l.* 12*s.* 8*d.* as 18*l.*, and not as 17*l.*

Contracted Multiplication of Decimals.

If we multiply a number containing four figures of decimals by another number containing four figures of decimals in the ordinary way, the product will contain eight decimal figures. Now decimals, as they occur in practice, are not exact but approximate quantities; and the use of four figures of decimals implies, speaking roughly, an uncertainty of five units either way in the fifth decimal place; that is to say, ·7854 may mean anything between ·78535 and ·78545. Similarly ·4142 may mean anything between ·41415 and ·41425. If we have to multiply ·7854 and ·4142 together, we get, in the ordinary way, ·32531268. But since these numbers are uncertain instead of exact, their product is uncertain also and may fall anywhere between

$$\cdot78535 \times \cdot41415 \text{ and } \cdot78545 \times \cdot41425$$

or between

$$\cdot3252527025 \text{ and } \cdot3253726625.$$

We can therefore only be certain of the product as far as these agree, viz. that it is ·3253 nearly. Hence, in order not

to throw away labour, it is desirable to use some method which will give us this product without superfluous results.

The contracted rule only differs from the ordinary rule in its arrangement, which enables us to discard the superfluous figures as they arise. The most convenient arrangement is to reverse the figures of the multiplier, so as to let the series of products extend to the right instead of the left; this makes no alteration in the product. For example:

```
            Common way              Altered process.
              7854                    7854
              4142                    2414
             -----                    ----
             15708                    31416
  (A)       31416          (B)          7854
            7854                        31416
           31416                          15708
           --------                   --------
           32531268                   32531268
```

Now if we want to retain five figures of the product, we can shorten the *altered process* by shortening each of the sub-products to the proper number of figures; thus:

```
            7854
            2414
            ----
            31416 . . . 4 × 7854
  (C)         785 . . . 1 × 785
              314 . . . 4 × 78   corrected
               16 . . . 2 × 7        ”

            32531
```

The actual process is not to write out (B) and shorten it, but is as follows: Multiply the first line by the first figure of the multiplier *simply*. For the second line (or any other) the first figure of the sub-product will be derived from the figure of the multiplicand, which stands *over the multiplying figure*. This rule gives the spacing right. Thus in the second line we begin with the 5 (the figure over the multiplier 1). When there are figures to the right of the figure we are using as multiplier, we must catch the carrying figure from the figure to the right. For example, in the third line, we have to begin 4 × 8 are 32; but there is a carrying figure from the 4 × 5, which makes this 34. We therefore put down the 4 and carry the 3. Then 4 × 7 + 3 = 31, which completes the line. The rest needs no explanation.

The object of reversing the figures of the multiplier is simply to get the product figures into their proper places with ease and certainty.

It is to be observed that the last figure, or even two figures, of the final product are not to be depended upon; we must therefore use a figure or two more than we require for the absolute limit of accuracy.

With regard to the number of figures to be retained, it is generally preferable to have regard rather to the total number of figures used than to the absolute number of decimals. Thus, if we multiply 15927·37 by ·54, where ·54 is an approximate and not an absolute figure, it is of no use to keep any decimals at all in the product. In fact, the result will only be good for the *thousands*. For the result can really be only known to lie somewhere between

15927·37 × ·535 and 15927·37 × ·545

that is, between

8521·14295 and 8680·41665.

Evidently it is idle to keep any decimals at all in such a case.

No general rule can be given for this. Practice and discretion must be relied upon for the extent to which it is safe to contract the work. If there is any doubt on the point, use a figure or two more.

As a second example of contracted multiplication take

1·9586991904 × ·5105429179		
19586991904		
9719245015		
97934959520		
1958699190		
97934960	. . . 5 × 19586991	corrected
7834797	. . . 4 × 1958699	"
391740	. . . 2 × 195869	"
176283	. . . 9 × 19586	"
1959	. . . 1 × 1958	"
1371	. . . 7 × 195	"
176	. . . 9 × 19	"
·9999999996		

For the decimal point, since $1{\cdot}9 \times {\cdot}5 = {\cdot}95$, we see that all the figures are decimals.

The determination of the unit figure or decimal point is practically effected at sight, without having recourse to any rule. A practised hand soon gets a sort of instinct as to where it ought to be, and persons who have not much practice will find no advantage in the contracted rules.

Contracted Division.

This is much less artificial than Contracted Multiplication. It will be best shown by an example. I shall give the work at full length and contracted.

Full Method.	Contracted Method.
7854) 27182818 (34610157	7854) 27182818 (34610157
23562	23562
36208	36208
31416	31416
47921	47921
47124	47124
7978	7978
7854	7854
12400	124
7854	79 . 1 × 78 cor.
45460	45
39270	39 · 5 × 7 „
61900	6
54978	5 or 6 . .
6922	(the carrying figure for 7 × 78)

The work of contraction, it will be seen, only amounts to cutting off the work, as it were, by a vertical line, but at the same time correcting what is left so as to do duty for the carrying or borrowing which belongs to the portion cut off.

In the above example the contraction begins at the re-

mainder 124. There is uncertainty on the last figure obtained, which should, in point of fact, be 9 and not 7.

I have shown all the products in the above example, but there is no more reason for inserting them than there is in ordinary division; thus:

```
7854 ) 27182818 ( 34610157
        36208
         47921
           7978
            124
             45
              6
```

If we had begun to contract one figure sooner, we should have had

```
7854 ) 2718282 ( 3461016
        36208
         47922
           798
           123
            44
```

Reciprocals.

The reciprocal of any number is the quotient of unity divided by that number. Thus the reciprocal of 13 is ·076923 or $\frac{1}{13}$. The reciprocal of ·5 is 2. The reciprocal of ·0125 is 80.

The use of reciprocals answers the same purpose in mixed multiplications and divisions as the arithmetical complement does in a mixture of additions and subtractions. For it is evident that multiplication by the reciprocal of a number is equivalent to division by the number itself; and

therefore, if we have a mixture of multipliers and divisors, we may take the reciprocals of the divisors and thus convert them into multipliers.

This is nothing more than, for example, multiplying by $\frac{4}{5}$ or ·8 instead of dividing by $\frac{5}{4}$ or 1·25. Large tables of reciprocals have been calculated. Those most used in this country are Barlow's. The tables of reciprocals are very useful in such cases as the following:

To find the decimal value of $\frac{365}{7988} + \frac{271}{5079}$.

The reciprocal of 7988 is ·0001251878.

The reciprocal of 5079 is ·0001968892.

Multiplying these by the numerators of the fractions and adding them, we have ·05375075 + ·04569355 = ·0994443.

Reduction of Money, Weights, and Measures to Decimals.

This is simply a question of multiplication or division. For example, suppose it be required to reduce 17*s.* 8½*d.* to the decimal of a pound.

1st. The ½*d.* expressed in pence is ·5. The pence are therefore 8·5. Now as each penny is $\frac{1}{12}$ of a shilling, the 8½*d.* expressed in shillings is $\frac{8·5}{12}$ = ·708333, &c. Hence the 17*s.* 8½*d.* may be written as 17·7083, or in pounds $\frac{17·7083}{20}$*l.* = ·8854167*l.* The sum is arranged as follows:

17*s.* 8½*d.* to be reduced to decimal of a pound.

```
 4 ) 2          farthings
12 ) 8·5
20 ) 17·708333  shillings
      ·8854167  pounds
```

Again, to reduce 11 cwt. 3 qrs. 22 lbs. 13 oz. to decimals of a ton:

```
 4 ) 13      oz.
 4 )  3·25
 4 ) 22·8125 lbs.
 7 )  5·703125
 4 )  3·814732 qrs.
20 ) 11·953683 cwt.
        ·5976841 tons.
```

The reduction from a decimal of a higher denomination to that of a lower denomination requires multiplication.

Thus, to reduce ·7854 of a square mile to acres, roods, poles, &c., we proceed as follows :

```
·7854 × 640                ·7854 × 640
   80                         80
62·832                     62·832
    8                          8
502·656 acres              502·656 acres
    4                          4
  2·624 roods              2010·624 roods
   40                         40
 24·96 poles               80424·96 poles
or 502 a. 2 r. 24·96 p.
```

The first sum is the way to find the a. r. p. in mixed denominations ; the second is the way to find the answer in poles only.

Conversion of Weights, Measures, and Values from One Unit to Another.

This is a question which presents itself in a great many different forms. The simplest case is that in which it is merely required to convert the measure of one country into

the corresponding measure of another. This requires us to be acquainted with the relation between the units of the two countries. For instance, suppose it be required to convert cwt., qrs., lbs., and oz. into French kilogrammes and their decimal subdivisions. We can do this if we know that a lb. is kilog. ·453593, by reducing the weight to English lbs. and decimals of a lb., and then multiplying by the factor of conversion. But if, instead of knowing the number of kilogs. in a lb. we had known the number of lbs. in a kilog., that is to say, lbs. 2·20462, we must have divided by this, which is the reciprocal of ·453593. For example, let the English weight be 3 tons 13 cwt. 2 qrs. 12 lbs. 13·48 oz. Reducing this to lbs. we have 8244·8425 lbs., and multiplying it by ·453593 we have 3740·71 kilogrammes.

A more complicated case, which presents itself quite as frequently, is of the kind illustrated by the following example:

Railway bars being 23 lbs. to the foot, to express the weight in kilogrammes per mètre.

This merely requires two applications of the same process. We first find how many pounds go to the mètre. For this purpose we multiply 23 by 3·2808992, the number of feet in a mètre. This gives 75·4606816 lbs. per mètre. This has now to be reduced to kilogrammes by multiplying by ·453593, giving 34·228437 kilogrammes per mètre.

As a further example of this kind of work, let it be asked how we shall convert 1,000 kilogrammes per cubic mètre into cubic feet per ton, having given as before 1 mètre =3·2808992 linear feet, and 1 kilogramme = 2·20462 lbs. We must first remember that a cubic mètre is a mètre in length, breadth, and thickness, and therefore the multiplier for converting mètres into feet must be used three times in succession. We shall therefore convert kilogrammes per cubic mètre into kilogrammes per cubic feet by dividing three times in succession by 3·2808992, and the multiplier for this will be

$$\overline{3{\cdot}2808992 \times 3{\cdot}2808992 \times 3{\cdot}2808992}$$

Hence the multiplier which will give us *pounds* per cubic foot will be

$$\frac{2{\cdot}20462}{\text{same den}^{\text{r}}.}$$

and the multiplier for tons per cubic feet will be this divided by 2240. Hence, 1,000 kilogrammes per mètre will be

$$\frac{1000 \times 2{\cdot}20462}{2240 \times 3{\cdot}28\ldots \times 3{\cdot}28\ldots \times 3{\cdot}28\ldots}$$

tons per cubic foot. But we want to find, not this, but the number of cubic feet to a ton, which will be its reciprocal—

$$\frac{2240 \times 3{\cdot}28\ldots \times 3{\cdot}28\ldots \times 3{\cdot}2808992}{1000 \times 2{\cdot}20462} = 35{\cdot}8825.$$

Approximate Decimalisation of English Money.

The true decimal division of the pound sterling ends with the florin or two-shilling piece, which is one-tenth of a pound; but since there are 96 farthings to the florin, the error made by taking odd farthings each as the 100th of a florin, or as ·001 of a pound is not very great. An easy correction enables us to get the decimalisation quite right as far as three places. The following is the rule. Decimalise the shillings at sight. Reduce the odd pence to farthings, and take these as decimals in the third place, adding 1 if there be odd threepence or more, and 2 if there be odd ninepence or more.

Thus for 17*s.* 7½*d.*, the shillings are at once written down as ·85, and the odd 7½*d.*=28 farthings, is written down as ·029; whence

17*s.* 7½*d.*=·879*l.*

The reason for the correction is that 3*d.* or 12 farthings is

$$\frac{\cdot 1}{8} = \cdot 0125 \text{ of a pound.}$$

If therefore the odd farthings be less than 12, the correction will be less than half a unit in the third place, and may be neglected; but if more, we must add one unit. Again, 9*d.* or 36 farthings is ·0375 of a pound, so we must add *one* from 3*d.* to 9*d.*, and *two* from 9*d.* up. Thus

17*s.* 2½*d* = ·860*l.*
17*s.* 7½*d.* = ·879*l.* not ·878*l.*
17*s.* 10½*d.* = ·894*l.* not ·892*l.*

To reduce from decimals of a pound to £ *s. d.*, get out the shillings first; then, if the odd decimals be less than ·013, take them as farthings; if between ·013 and ·038, subtract 1 and take as farthings; if more than ·038, subtract 2 and take as farthings. Thus

·561*l.* = 11*s.* and ·011*l.* = 11*s.* 2¾*d.*
·723*l.* = 14*s.* and ·023*l.* = 14*s.* 5½*d.*
·441*l.* = 8*s.* and ·041*l.* = 8*s.* 9¾*d.*

This rule is exceedingly useful to persons who have many approximate money calculations to make. For instance, the question of how much per cent. is 13*s.* 7½*d.* in the pound is solved at once by it.

13*s.* 7½*d.* = ·65 + ·031*l.*
= ·681*l.*
= 68·1 per cent.

Again, 79·7 per cent. is ·797 of a pound
= 15*s.* + ·047*l.* = 15*s.* 11¼*d.*

Those who need this rule will find no difficulty in procuring examples, which they can verify by actual decimal reduction, until they are expert in its use.

Many similar tricks may be invented for rough reductions. Thus, English linear measure may be reduced to French by

reducing to yards and subtracting 10 per cent, or $\frac{1}{10}$, and calling the remainder mètres. To reduce mètres to English yards, add 10 per cent. This, however, is very rough.

For rough purposes, again, a ton may be considered as 1,000 kilogrammes. Hence arises an easy rule for comparing English and French weights.

Decimal Fractions.

1. Express as vulgar fractions and reduce to the lowest terms:

(1) ·5.
(2) ·025.
(3) ·625.
(4) 6·75.
(5) ·00032.
(6) ·0000001024.
(7) ·00256.
(8) ·00000000576.
(9) ·12.
(10) ·0135.
(11) ·0001875.
(12) ·725.
(13) ·576.
(14) 1·68.
(15) 1·024.
(16) ·9241.
(17) 67·09.
(18) ·006693.
(19) ·2064919.
(20) ·00000000000000065536.

2. Add together

(1) 375, ·0049, 37·24, 5·93. 67·489, ·5, and 60.
(2) ·3, ·03, ·003, 30, 300, ·0003, and 3.
(3) 693·894, 23·71, 2·7657, ·008, ·8594, and 52·5.
(4) 20·07, 18·3, 1·45, 2·00345, ·8, ·15, 51·7, and 187.
(5) 3·14159, 51·04, 2·3725, ·008, 471·7, 83·954, and 839·54.
(6) 2·71828, 31·416, ·0094, 68·584, 6·28172, 1·9906, and ·11111.
(7) Add the totals.

3. Find the result of the following mixed additions and subtractions:

(1) 18·79 + 1·003 − 19 + 271·94 − ·000125.
(2) 243·12 + ·0101 − 79·89 + 8·451 − 129·99.

(3) $9{\cdot}745 + 5{\cdot}079 - 291{\cdot}45 + 27{\cdot}834 + 300$.

(4) $19{\cdot}123 - 20{\cdot}9781 + 35{\cdot}001 - 49{\cdot}775 + 16{\cdot}175$.

(5) $1{\cdot}09581 - 36{\cdot}2 + 11{\cdot}57 - 25{\cdot}0005 + 29{\cdot}875$.

(6) ${\cdot}30103 + {\cdot}47712 - {\cdot}84510 + {\cdot}22185 - {\cdot}15490$.

(7) Add the results.

4. Examples of multiplication :

(1) $2{\cdot}5 \times 4$.

(2) ${\cdot}25 \times 40$.

(3) $2{\cdot}5 \times 476$.

(4) $2{\cdot}5 \times 4{\cdot}76$.

(5) ${\cdot}0025 \times 4{\cdot}76$.

(6) ${\cdot}025 \times {\cdot}0476$.

(7) $1{\cdot}00039 \times {\cdot}09996$.

(8) ${\cdot}128 \times {\cdot}0078125$.

(9) $1950{\cdot}625 \times {\cdot}0001152$.

(10) ${\cdot}00044893 \times {\cdot}002227519$.

5. Division : express decimally

(1) $\frac{17}{32}$.

(2) $\frac{19}{625}$.

(3) $\frac{125}{1024}$.

(4) $\frac{32}{78125}$.

(5) $\frac{1001}{16384}$.

(6) $\frac{3{\cdot}75}{{\cdot}01024}$.

(7) $\frac{{\cdot}01137}{{\cdot}00625}$.

(8) $\frac{1}{1048{\cdot}576}$.

(9) $\frac{{\cdot}001024}{30517578125}$.

(10) $\frac{4000}{{\cdot}000125}$.

(11) $\frac{1000000}{{\cdot}0000001}$.

(12) $\frac{3{\cdot}2}{781{\cdot}25}$.

All the above are terminating decimals.

6. Further examples of division :

(1) $\frac{1{\cdot}5}{17}$ to 10 decimal places.

(2) $\frac{18{\cdot}1}{{\cdot}27}$ to 5 decimals.

(3) $\frac{19{\cdot}5}{1130}$ to 12 decimals.

(4) $\frac{{\cdot}001}{10{\cdot}01}$ to 25 decimals.

(5) $\frac{{\cdot}0001}{{\cdot}1111}$ to 25 decimals.

(6) $\dfrac{\cdot 06561}{531441}$ to 15 decimals.

(7) $\dfrac{1}{\cdot 0011636}$ to 2 decimals.

(8) $\dfrac{}{859\cdot 43689}$ to 6 decimals.

(9) $\dfrac{\cdot 9921461}{\cdot 0078539}$ to 6 decimals.

(10) $\dfrac{\cdot 0078539}{\cdot 9921461}$ to 6 decimals.

(11) $\dfrac{159\cdot 982}{\cdot 00098400018}$ to 7 decimals.

(12) $\dfrac{\cdot 98400018}{\cdot 00159982}$ to 6 decimals.

7. Find the circulating periods arising out of the following divisions:

(1) $\frac{1}{3}$. (2) $\frac{1}{7}$. (3) $\frac{5}{7}$. (4) $\frac{4}{9}$. (5) $\frac{5}{9}$. (6) $\frac{1}{9}$. (7) $\frac{1}{11}$. (8) $\frac{7}{11}$. (9) $\frac{10}{11}$. (10) $\frac{1}{13}$. (11) $\frac{2}{13}$. (12) $\frac{11}{73}$. (13) $\frac{5}{117}$. (14) $\frac{10}{481}$. (15) $\frac{41}{492}$. (16) $\frac{1}{9091}$. (17) $\frac{61}{4649}$. (18) $\frac{17}{9901}$.

8. Reduce the following pure circulates to vulgar fractions in their lowest terms:

(1) $\cdot\dot{9}6$.
(2) $\cdot\dot{1}4285\dot{7}$.
(3) $\cdot\dot{4}2857\dot{1}$.
(4) $\cdot\dot{0}2564\dot{1}$.
(5) $\cdot\dot{0}5128\dot{2}$.
(6) $\cdot\dot{9}7435\dot{8}$.
(7) $\cdot\dot{9}4871\dot{7}$.
(8) $\cdot\dot{0}\dot{1}$.
(9) $\dot{8}\dot{3}$.
(10) $\cdot\dot{8}$.
(11) $\cdot\dot{2}335766\dot{4}$.
(12) $\cdot\dot{0}00215\dot{1}$.
(13) $\cdot\dot{0}00100999899\dot{9}$.
(14) $\cdot\dot{0}58823529411764\dot{7}$.
(15) $\cdot\dot{0}34482758620689655172413793\dot{1}$.

9. Reduce the following mixed circulates to vulgar fractions in their lowest terms:

(1) $\cdot 31\dot{6}$.	(6) $8\cdot 4\dot{8}\dot{4}$.
(2) $\cdot 1\dot{3}\dot{6}$.	(7) $\cdot 9\dot{1}4285\dot{7}$.
(3) $\cdot 1\dot{6}\dot{3}$.	(8) $\cdot 7\dot{0}0245\dot{7}$.
(4) $\cdot 577\dot{5}$.	(9) $\cdot 29975\dot{4}\dot{2}$.
(5) $\cdot 59\dot{2}\dot{5}$.	(10) $\cdot 75\dot{9}972451790633608815 42\dot{6}$.

10. Find the approximate values of the following series to 7 decimal places:

(1) $1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \frac{1}{1.2.3.4.5}$ + &c.[1]

(2) $1 - 1 + \frac{1}{1.2} - \frac{1}{1.2.3} + \frac{1}{1.2.3.4} - \frac{1}{1.2.3.4.5}$ + &c.

Show that (2) is the reciprocal of (1).

(3) $1 - \frac{1}{1.2.3} + \frac{1}{1.2.3.4.5} - \frac{1}{1.2.3.\ldots.7} + \frac{1}{1.2.3.\ldots.9}$ − &c.

(4) $\frac{1}{1.2} - \frac{1}{1.2.3.4} + \frac{1}{1.2.3.4.5.6} - \frac{1}{1.2.3.\ldots.8} + \frac{1}{1.2.3.\ldots.10}$ − &c.

(5) $\frac{1}{2} + \frac{1}{6.8} + \frac{1}{24.32} + \frac{61}{5040.128} + \frac{277}{72576.512}$

(6) $\frac{1}{2} - \frac{1}{120} + \frac{1}{2400} - \frac{19}{720000} + \frac{3}{1600000} - \frac{863}{6048000000}$

[1] As already stated, a stop or point at the foot of the line between two figures is the sign of multiplication. Thus, 2 . 3 means 2×3 or 6; 1 . 2 . 3 . 4 . 5 means $1 \times 2 \times 3 \times 4 \times 5 = 120$. The terms of the first four series are constructed by an evident law. As many terms must be taken as may be necessary to give the requisite number of figures.

11. Contraction of decimals.

(1) Cut off 1, 2, 3, 4, 5, or 6 figures from 3·14159162.

Answer. 3·1415916, 3·141592, 3·14159, 3·1416, 3·142, 3·14.

Cut off figures in succession from

(2) 2·718281828.
(3) 2·302585092994.
(4) ·301029995664.
(5) ·434294481903.
(6) ·577215664901532860606 5.

12. Examples for contracted multiplication and division.

$$·47712 \times 2·095909 = 1.$$
$$·56789 \times 1·760904 = 1.$$
$$·456789 \times 2·189185 = 1.$$
$$65·8710862 \times 65·8710862 = 4339.$$
$$88·6171541 \times 88·6171541 = 7853.$$
$$20·166936 \times 20·166936 \times 20·166936 = 8202.$$
$$·0874887 \times ·9961947 = ·0871557.$$
$$·0871557 \times 11·430052 = ·9961947.$$
$$·434294481903 \times 2·302585092994 = 1.$$
$$2·718281828459 \times ·367879441171 = 1.$$

These will serve to give examples of division as well as of multiplication, and they may be varied by cutting off some of the figures according to the prescribed rule. The examples in groups 4, 5, and 6 may also be worked by the contracted method.

13.

(1) Reduce the half-crown, shilling, threepenny piece, penny, and farthing, to decimals of a pound.

(2) Reduce the cwt., qr., lb., and oz. avoirdupois to decimals of the ton; the qr., lb., and oz. to decimals of a cwt., &c.

(3) Reduce the lower denominations to decimals of the higher ones in *square measure*—square miles, acres, roods, chains, poles, links, yards, feet, and inches.

(4) Reduce degrees, minutes, and seconds to decimals of the circumference of the circle.

(5) Taking the semi-circumference as 3·1415926536 (radius taken as unity), what are the lengths of the degree, minute, and second?

(6) How many degrees (and decimals) are there in the arc equal to the radius?

14. Express in decimals of a pound:

(1) 17*s.* 6*d.*	(3) 9*s.* $5\frac{3}{4}$*d.*	(5) 5*l.* 11*s.* $7\frac{1}{4}$*d.*
(2) 16*s.* 8*d.*	(4) 3*l.* 19*s.* $11\frac{3}{4}$*d.*	(6) 4*l.* 13*s.* $9\frac{1}{2}$*d.*

15.

(1) Convert 37·8593 mètres into English measure.

(2) Convert 3 miles 1352 yards into French measure.

(3) Convert 18 tons 13 cwt. 2 qrs. 21 lbs. into French weight.

(4) Convert 3·597832 kilogrammes into avoirdupois grains.

(5) Convert 3 oz. 5 dwt. 13 gr. into grammes.

(6) Convert 14 sq. miles 78 acres into hectares.

(7) Convert 395·47 acres into a., r., p.

(8) What is the measure of a litre in English cubic inches?

(9) What is the weight of a gallon of fresh water in kilogrammes?

(10) Reduce 2·8735 cubic mètres to English feet and inches.

16.

(1) Express 14 cwt. 3 qrs. 17 lbs. 5 oz. in decimals of a ton.

(2) Express 13 oz. 5 drams 7 scruples 19 grs. in decimals of a pound and of an ounce.

(3) Express 57° 17′ 44·8″ in terms of the semi-circumference and of the radius.

(4) Express 4 sq. m. 347 a. 3 r. 47 p. in terms of a square mile and of an acre.

(5) Express 19 cubic yards 21 ft. 1397 inches in tons of 35 cubic feet.

(6) Express 12 lbs. 7 oz. 6 dwts. 11 gr. troy in decimals of an avoirdupois pound.

(7) Express 17 lbs. 15 oz. 400 gr. avoirdupois in decimals of the troy pound.

(8) Express 57 a. 2 r. 37 p. customary measure of 6 linear yards to the pole, in statute acres and decimals.

17. Reductions to lower denominations:

(1) 3·14159265 tons to cwt. qrs. lbs. oz., and to ounces only. Also to grains troy.

(2) ·318309886 of the semi-circumference of a circle to degrees, minutes, and seconds, and also to seconds.

(3) ·43428793 of a year (of 365 days) in days, hours, &c.

(4) ·57721314987 of a sq. mile in acres and decimals, and in feet and decimals.

(5) ·487956324 tons of sea-water in cubic feet and inches (35 ft. = 1 ton).

(6) Express the same in gallons of 277·274 cubic inches.

18.

(1) What multiplier will convert lbs. per foot run into kilogrammes per mètre?

(2) What multiplier will convert kilogrammes per mètre into lbs. per foot run?

(3) What multiplier will convert lbs. to the cubic foot into kilogrammes per cubic mètre?

(4) What multiplier will convert kilogrammes per square mètre (say of sheet lead of uniform thickness) to lbs. to the square foot?

(5) What multiplier will convert cubic mètres to the kilogramme into inches to the troy grain?

(6) What multiplier will convert circular inches into square feet?

(7) A cubic foot weighs 445 lbs. What does a pound measure in inches?

(8) What is the decimal rate per cent. of 7*d.* in the pound?

(9) A gallon of water weighs 10 lbs. and measures 277·274 cubic inches. What is the weight of a cubic foot, and the measure of a ton?

(10) What is the weight of a gallon of sea-water reckoned at 35 feet to the ton?

(11) What multiplier will convert tons to the cubic yard into ounces per cubic inch?

(12) What multiplier will convert grains troy to the cubic inch into avoirdupois pounds per foot?

CHAPTER XIV.

MISCELLANEOUS RULES.

Averages and Means.

A mean in its most general sense implies some number intermediate to other numbers; but the arithmetical mean has a more definite meaning, and is found by adding all the numbers together, and dividing by the number of items. Thus the mean of the two numbers 7 and 13 is $\frac{7+13}{2} = 10$. The mean of the three numbers, 14, 17, and 26 is $\frac{14+17+26}{3}$ $= \frac{57}{3} = 19$.

An average, in the sense in which it is used in statistics, does not differ in any respect from the arithmetical mean, although its form is occasionally a little disguised. For instance if there be 14 collections of units, and if the total number of units in all the collections be 2,800, the average per collection is 200. We must here call attention to a principle which we shall revert to in the case of percentages: 'A mean of a set of averages is not, in general, the same as the gross average.' Take, for instance, six groups of schools with the total attendances given in the third column of the

following table. We obtain the materials for the fourth column by dividing each entry in the third by the corresponding entry in the second.

Group	Number of Schools	Number of Children	Average
1	17	1,700	100
2	10	475	47·5
3	29	7,431	256·2
4	8	240	30
5	35	11,925	340·7
6	12	750	62·5
Total .	111	22,521	203

The 203 is obtained from $\frac{22,521}{111}$. It is therefore the gross average. The mean of the averages is $\frac{836\cdot9}{6}=139\cdot5$, which is a very different thing.

There is another sense in which the word mean or average is used, and that is when it is applied to a variable quantity. Take, for instance, the case of a pond or lake with an uneven bottom and suppose the question asked What is the average depth? To settle what is meant by this, we must remark that an average answers the question, 'If all the lots were equalised what would each lot contain?' when we are speaking of a definite number of lots, each containing a specified number of articles. If we apply this definition of an average to the question of the pond, the average depth simply answers the question; suppose we dug another pond of the same shape and size, but of uniform depth, how deep must it be to hold the same quantity of water?

So much for the meaning of average depth. In order to find it numerically, I remark that we know from geometry that in a lake of uniform depth the surface in square feet multiplied by the depth in linear feet will give the contents in cubic feet. Hence, if we can measure the quantity in any

such pond in cubic feet, and the upper surface in square feet the average depth is the fraction $\frac{\text{contents}}{\text{surface}}$.

The determination of the contents and surface falls under MENSURATION.

A simpler case than this is the average breadth of a plane surface. The average breadth evidently answers this question: 'What is the breadth of a strip of the same length, but of uniform breadth, having the same area?' Since the area of the uniform strip is length × breadth, this uniform breadth or mean breadth of the original piece will be $\frac{\text{area}}{\text{length}}$. Thus if the surface of a sheet of water be 97 square miles and its length 19 miles, its mean breadth will be $\frac{97}{19} = 5{\cdot}10526$ whatever the shape may be.

Percentages.

Percentage is a device to escape the use of fractions. Thus, if a quarter of the men in a regiment are in hospital we say that 25 per cent. of them are in hospital. It is a mere circumlocution for ·25. It is sometimes expressed as follows: Out of every 100 men 25 are in hospital, but this, to say the least, gives a peculiar meaning to the term 'every.' What it means is that the number in hospital would be the same if you divided the regiment into companies of 100 each and then sent 25 of each company to the hospital. In talking of a percentage in this way we lose sight both of the individual soldier and of the numerical strength of the regiment. In dealing with percentages it is very important to remember this.

A comparison of percentages is merely a comparison of proportions. If one regiment has 20 per cent. in hospital and a second has 25 per cent., we may say that the first is *proportionately* healthier than the second, but we have no right to infer that it has more men out of hospital or fewer men in hospital than the second. To arrive at this we must know the comparative strength of the two regiments. Further

if these regiments are brigaded together, we cannot infer the sickness of the brigade per cent. from the separate percentages of the two regiments which compose the brigade. For this purpose also we require to know the strengths of the regiments. To show this let us first assume that

Regiment A, 1,000 strong, has 20 per cent. sick.
Regiment B, 500 strong, has 25 per cent. sick.

The actual number sick in regiment A will be 200; in regiment B 125. In the brigade therefore there will be 325 sick out of 1500 men. The percentage of sick therefore will be $\frac{325 \times 100}{1500}$, or $\frac{65}{3}$, or $21\frac{2}{3}$ per cent. Next, let us reverse things and suppose that A is 500 strong and B 1,000 strong. Then the number of sick in A will be 100 and in B 250, or 350 in all, out of 1,500 as before. Hence the percentage will now be $\frac{350 \times 100}{1,500}$ or $\frac{70}{3}$ or $23\frac{1}{3}$ per cent., which is a very different result from what was formerly got. It is not necessary, however, that we should know the actual strength of the regiments; it is sufficient if we know their relative strengths. Suppose, for instance, that the regiments are known to be in the proportion of 4 to 3; i. e., that B is $\frac{3}{4}$ of the numerical strength of A. This is equivalent to supposing that A consists of four companies, and B of three. Let us now take for our unit $\frac{1}{100}$th part of a company. It does not matter whether any such unit physically exists or not, that is to say, since we are going to lose sight of the unit, it does not matter whether the unit is half a man or a whole man. This settled, regiment A will have 80 units sick, and regiment B 75, or 155 out of 700 units. The percentage of this is $\frac{155}{7} = 22\frac{1}{7}$.

The rule for combining percentages is this—If the numbers be in the proportion of A to B and the corresponding percentages of a to b, the gross percentage will be $\frac{A \times a + B \times b}{A + B}$.

Thus in the case just stated A is 4 and B is 3; a is 20 and b 25. The gross percentage is

$$\frac{4 \times 20 + 3 \times 25}{4 + 3}, \text{ or } \frac{155}{7} \text{ as before.}$$

A similar rule applies to any number of percentages. Let the absolute numbers be in the proportion, A, B, C, D, &c., and let the percentages be a, b, c, d, &c. Then the gross percentage will be

$$\frac{A \times a + B \times b + C \times c + \&c.}{A + B + C + \&c.}$$

The only reason that I can see for using percentages at all is to suit the ignorance of people who are frightened by the name of decimal fractions. They are simply vulgar fractions reduced to the common denominator 100. For all practical purposes the absolute proportion to unity is much more useful. It follows the same rules and is less confusing. Percentages, however, are still used in most statistical returns, where there are people or things. They are also used commercially in the statement of profits and losses, interests, discounts, and commissions.

A percentage upon a percentage is a mere question of multiplication. If, for instance, we say that 25 per cent. fall sick, and 40 per cent. of the sick die, the percentage of deaths is $\frac{25}{100} \times \frac{40}{100} = \frac{1,000}{10,000} = \frac{10}{100}$ or 10 per cent.

If we had used the absolute proportions instead of the percentages we should have escaped all trouble about the denominators and simply have written $\cdot 25 \times \cdot 4 = \cdot 1$.

Commissions.

These are generally stated at so much per cent. Their calculation is simply the working of the rule of three sum:

$$100 : \left\{ \begin{matrix} \text{commission} \\ \text{per cent.} \end{matrix} \right\} :: \text{cost} : \text{commission}.$$

Thus the commission on 225*l.* at 5 per cent. is 11*l.* 5*s.*

Examples.

1. What is the mean of

(1) 7 and 18. (2) 8, 19, and 53.
(3) $\frac{3}{4}$ and $\frac{4}{3}$. (4) $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$.
(5) 0, 0·30103, 0·47712, 0·60206, 0·69897, 0·77815, 0·84510, 0·90309, 0·95424, 1.
(6) 0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100.
(7) 0, 1, 1·4142, 1·7321, 2, 2·2361, 2·4495.
(8) 0, 1, 1·2599, 1·4422, 1·5874, 1·7080, 1·8171, 1·9129.

2. The following summary is taken from a sale book :

April	3	sold	250	@ $10\frac{1}{2}d.$	each.
„	4	„	1,000	@ 9*d.*	„
„	5	„	500	@ $9\frac{1}{2}d.$	„
„	6	„	3,000	@ 9*d.*	„
„	7	„	1,500	@ 11*d.*	„
„	8	„	375	@ 10*d.*	„

What are the average selling price (using decimals of a penny), the average number sold daily, and the average daily cash business of the week?

3. Fill up the fourth and fifth columns of the following table :

Name of Town	Population in 1861	Population in 1871	Increase per cent.	Decrease per cent.
A	35,134	33,007		
B	192,286	190,405		
C	283,721	321,182		
D	94,876	92,024		
E	77,520	78,110		
F	43,791	39,348		
G	24,248	22,907		
H	62,707	58,421		
Total .				

4. Find the mean percentage and the gross average percentage from the following results of an examination in arithmetic.[1]

Place	Number presented for examination	Percentage of candidates who obtained marks of	
		'Good'	'Fair'
Bangor	14	14·29	64·28
Battersea	39	23·07	64·10
Carmarthen	22	22·73	68·18
Carnarvon	11	9·09	72·72
Chelsea	55	30·90	65·45
Cheltenham	34	29·41	58·82
Mean of percentages			
Gross percentage			

5. A ditch runs alongside of a straight road, but at a little distance from it, for 100 yards. The exact distance of the edge of the ditch from the road is measured at every 10 yards and these measurements are found to be 1, 1·5, 2, 2·2, 2·5, 2·75, 3·5, 3·75, 4, 3·5, 2·5 in yards. What is the mean distance of the edge of the ditch, and the approximate area of the piece of waste between the ditch and the road?

6. How much per cent. do the following poundages amount to?

[1] This is taken from an actual return in a blue book. It shows an abuse of figures which is not uncommon, and against which it is useful to caution students. The carrying out the percentage to decimals, when the number of cases is so small, is a mere absurdity; for it is evident that one of the men under examination having a headache and blundering over a sum might alter the unit figure as well as the decimals. Many statistics thus make a show of accuracy quite unwarranted by the facts.

(1) 9*d.* (2) 7*d.* (3) 6*d.* (4) 1*s.* 6*d.*
(5) 6*s.* 8*d.* (6) 7*s.* 6*d.* (7) 16*s.* 8*d.* (8) $3\frac{3}{4}$*d.*

7. The percentage of sick in certain garrisons in France, Italy, and India, and the strength of the garrisons being as follows :

France		Italy		India	
Strength	Sick per cent.	Strength	Sick per cent.	Strength	Sick per cent.
5,934	11·5	4,850	14·3	7,550	23·2
11,220	9·2	5,792	16·1	9,475	17·4
2,339	18·4	3,720	8·2	3,521	8·3
7,954	14	2,595	23	9,728	7·8
4,789	13·7	7,482	12·7	8,240	7·9
9,375	8	2,235	6	7,667	8·1

what is the percentage of sickness in each group and altogether?

8. Of a certain army 65 per cent. are at home and the rest abroad. Of the home army 10 per cent. are artillery, 15 per cent. are cavalry, and the rest infantry. Of the foreign army 12 per cent. are artillery and 18 per cent. cavalry. What percentage of the whole artillery is serving abroad, what of the cavalry, and what of the infantry?

Simple Interest, Discount, and Stocks.

Interest is a payment made by a borrower for the use of money. It is usually quoted as so much per cent. per annum, that is to say, so much for the use of 100*l.* for one year. Simple Interest is reckoned proportionately for any sum of money differing from 100*l.* or for any other period than a year. Thus if it be desired to find the interest of 220*l.* for $2\frac{1}{2}$ years at 5 per cent. we may either do it by finding the interest on 220*l.* for one year and then for $2\frac{1}{2}$ years, or we may find the interest for 100*l.* for $2\frac{1}{2}$ years and thence for 220*l.* In the first case the proportions run—

As 100 : 5 :: 220*l.* : 11*l.*;

then.

As 1 : $2\frac{1}{2}$ :: 11 : 27*l.* 10*s.*

In the other way it would run—

As 1 : 5 :: $2\frac{1}{2}$: 12*l.* 10*s.*;

then,

As 100 : 12*l.* 10*s.* :: 220*l.* : 27*l.* 10*s.*

The work may be more shortly put down as follows:

$$5 \times 2\frac{1}{2} \times \frac{220}{100} = 27\frac{1}{2}l. = 27l.\ 10s.$$

This is Simple Interest.

Compound Interest.

If the interest, instead of being paid when due, be left in the borrower's hands, it is reasonable that he should pay for this accommodation in just the same way as if he had been obliged to have recourse to some third person to lend him this interest. When money is lent upon these terms it is said to be lent upon compound interest, that is to say, interest being charged upon interest. For example, suppose a person to have borrowed 200*l.* at 5 per cent. compound interest. At the end of the first year 10*l.* of interest falls due, and from that date he may be considered to owe the lender 210*l.* The interest on this at 5 per cent. is 10*l.* 10*s.* From and after the end of the second year the borrower will owe 220*l.* 10*s.* The interest on this for a year is 11*l.* 0*s.* 6*d.*, therefore after the end of the third year the borrower will owe 231*l.* 10*s.* 6*d.* The calculation may thus be carried on for as many years as may be necessary. In this sort of calculation it is better to consider, not the rate per cent., but the decimal rate of interest charged for 1*l.* for a year. In the case of 5 per cent. this is ·05. The amount at the end of the first year will be the original sum in pounds × 1·05. The amount at the end of the second year will be this again × 1·05, and so on.

Those who are much in the habit of making calculations

at compound interest use tables in which the amount of 1*l.* lent at a given rate of compound interest for so many years is calculated. In the case of 5 per cent. the first entry in the table is 1·05, the second $1·05 \times 1·05$, or 1·025, the third entry is $1·05 \times 1·05 \times 1·05$ or 1·157625, and so on. To find the amount of any sum whatever at this rate of interest is then simple multiplication.

Compound interest presents certain difficulties which cannot be explained without a knowledge of algebra. It is worth while to remark that the steady increase and decrease of population follow the law of compound interest more nearly than that of simple interest.

It is usual to call the sum of money on which the compound interest is reckoned, the *original amount* or *principal.* The principal and interest together, after any number of years, is called the *amount at the end* of so many *years.* Thus 105*l.* is the amount of 100*l.* at the end of one year, at 5 per cent., whether at simple or compound interest.

The *present value* of a sum payable a number of years hence, at a certain rate of interest, is evidently got by using as a divisor the same figure which, if used as a multiplier, would give the amount of that sum after the same number of years. Thus the amount of 400*l.* at the end of two years at 5 per cent. is $400 \times 1·05 \times 1·05$, or $400 \times 1·1025$, or 441*l.* 2*s.*, while the present value of 400*l.* payable two years hence is—

$$\frac{400}{1·1025} = 362·82l. = 362l.\ 15s.\ 5d.$$

The *present value* of one pound is the reciprocal of its *amount*, the period being the same in both cases.

Discount.

Discount is the abatement made in consequence of money being paid before it is due, the value of this accommodation being reckoned in the same manner as simple interest. There is here a difference between *true discount* and the

discount usually allowed in commercial transactions. The difference between the two will be best seen by taking a simple example. Suppose a person is under obligation to pay me a hundred pounds this day twelvemonth, but that I want the money at once and am willing to pay 5 per cent. interest for this advantage. Practically, a banker would simply deduct the interest for a year and pay me 95*l.* instead of 100*l.* But in this case I am made to pay 5*l.* interest for a loan of 95*l.*, and this is evidently more than 5 per cent. interest. The *true discount* is found by means of the following question: What sum laid out at 5 per cent. interest will amount to 100*l.* at the end of the year? This is simply the proportion sum. As 105 : 100 :: 100 : amount which I ought to receive, which is therefore $\frac{10,000}{105}l.$ $= 95{\cdot}2381l.$, or 95*l.* 4*s.* 10*d.* very nearly. Hence the true discount will be 4*l.* 15*s.* 2*d.*

The difference between true and commercial discount is not very great when the period is short and the rate of interest moderate. We have just found that at 5 per cent. the difference was 4*s.* 10*d.* on 100*l.* But there are certain classes of bill transactions in which the difference is very serious. Thus in a bill for 12 months the true discount at 50 per cent. is 33*l.* 6*s.* 8*d.*, while the practical discount is 50*l.*, which is equivalent to charging 100 per cent. for the use of the money seeing that the borrower receives 50*l.* now for which he contracts to pay 100*l.* at the end of the year.

In strictness discounts for periods other than a whole year ought to be true discount taken at compound interest, that is to say, the difference between the principal sum and the *present value.* It is upon this principle that the actual values of deferred payments are calculated by large speculators and Life Assurance Companies. But this requires a knowledge of algebra, or at least of logarithms.

Stocks.

A Government Stock really means an undertaking on the part of Government to pay a fixed sum annually. The price of the Stock is simply the actual value of this annuity, a value which depends partly upon reliance on the ability and willingness of the Government to meet its liabilities and partly on the current demand for ready money. Instead of quoting it as an annuity of so much a year, it is usual to consider it as representing the interest on an amount nominally or actually lent to the Government: and the Government generally retains the right of redeeming it at this rate. Thus 100*l.* 3 per cent. stock means simply an undertaking on the part of the Government to pay 3*l.* annually to the person holding the stock, or to any person to whom he chooses to transfer it, with a power of redeeming this pledge on payment of 100*l.* This 100*l.* has, of course, very little to do with the value. When 'English 3 per cents. are at 91,' it simply means that a perpetual annuity of 3*l.* a year guaranteed by the English Government is worth 91*l.* This is called 100*l.* stock. But the reference to the 100*l.* is very little more than a convenient fiction. The following are examples of some of the arithmetical questions that arise concerning stocks:

1. 5,000*l.* are invested in the 3 per cents. at 91. How much stock must be bought? What would be the half-yearly dividend, and how much per cent. per annum would it return as an investment?

The quantity of stock bought must be $5,000 \times \frac{100}{91}$, or 5,494*l.* 10*s.* nearly. The annual dividend upon this is found, by multiplying it by $\frac{3}{100}$, to be 164*l.* 16*s.* 8½*d.*

	£	s.	d.
	5,494	10	0
			3
10)	16,483	10	0
10)	1,648	7	0
	164	16	$8\frac{1}{2}$

The dividend is therefore 82*l.* 8*s.* 4*d.*

The percentage which it returns as an investment is $\frac{164l.\ 16s.\ 8\frac{1}{2}d.}{50}$ $=3l.\ 3s.\ 11d.$, being obtained from the proportion

5,000 : 164*l.* 16*s.* $8\frac{1}{2}$*d.* :: 100 : rate per cent.

2. I invest in the $3\frac{1}{2}$ per cents. at 97, and after 3 years I sell at 104. How much per cent. have I made of my capital?

The actual amount invested is immaterial in this question. For convenience suppose it to be 100*l.* The quantity of stock which this purchased was $100l. \times \frac{100}{97}$. The income was therefore $100 \times \frac{3\frac{1}{2}}{97}$, and this I took for three years, making altogether $100 \times \frac{10\frac{1}{2}}{97}$. I then sold it for $104 \times \frac{100}{97}$. The dividends and profits for the three years are therefore—

$$100 \times \frac{104}{97} - 100 + 100 \times \frac{10\frac{1}{2}}{97}$$

$$\text{or } 100 \times \frac{7}{97} + 100 \times \frac{10\frac{1}{2}}{97}$$

$$\text{or } 100 \times \frac{17\frac{1}{2}}{97} = \frac{1,750}{97} = 18\cdot041l.$$

$$= 18l.\ 0s.\ 10d.$$

And this is the proceeds of 100*l.* invested during three years. Hence the investment has paid about 6*l.* per cent. per annum.

3. I sell 6,000*l.* stock out of English 3 per cents. at 91 and I invest in Dutch 5 per cents. at 120. By how much do I increase or diminish my income?

The English stock returns an income of 180*l.* per annum. It sells

for 91*l.* × 60 = 5,460*l.* This buys $\frac{5,460 \times 100}{120}$ = 4,550*l.* Dutch 5 per cents., the income of which is 227*l.* 10*s.* I therefore increase my income by 47*l.* 10*s.*

Examples.

1. Find the simple interest of

(1) 375*l.* at $3\frac{1}{2}$ per cent. for a year.

(2) 5,191*l.* at 16*l.* 13*s.* 4*d.* per cent. for a year.

(3) 1,000*l.* at $4\frac{1}{2}$ per cent. for $5\frac{1}{2}$ years.

(4) 100*l.* at $5\frac{1}{4}$ per cent. for 1 year and 198 days.

(5) 4,728*l.* at $2\frac{1}{2}$ per cent. for 2 years and 219 days.

(6) 5,9874*l.* at 3*l.* 18*s.* 4*d.* per cent. for 3 years and 240 days.

2 (1) What sum at $3\frac{1}{2}$ per cent. will pay 100*l.* a year?

(2) 3,700*l.* was invested and paid 740*l.* in 3 years and 6 months. What was the rate of interest per cent. per annum?

(3) 5,000*l.* invested at $4\frac{1}{2}$ per cent. paid 1,000*l.* in all? For what period?

(4) At $3\frac{1}{2}$ per cent. per annum, the interest for 5 years and 3 months came to 7,000*l.* What was the capital?

✔(5) A banker borrowed some money at 3*l.* 6*s.* 8*d.* per cent., and lent it at 5 per cent. After seven years he found he had cleared 10,000*l.* What was the sum borrowed?

(6) A gentleman borrowed 500*l.* for 6 months at the rate of 10 per cent. per annum, and his legal expenses came to 7*l.* 10*s.* What rate of interest did he actually pay out of pocket?

(7) A poor woman borrowed 20*l.* which she repaid by eleven monthly instalments of 2*l.* each, the first being taken on the day of borrowing. What was the annual rate of interest (reckoned as simple interest)?

(8) Money was lent on bad security at 25*l.* per cent. per annum. Six years' interest was paid, and then the capital was lost. As a question of simple interest, how much did it pay?

3. (1) When the 3 per cents. are at 89, what is the rate of interest?

(2) What is the half-yearly dividend on 1,375*l.* invested in French fives at 56?

(3) What sum must I invest in $3\frac{1}{2}$ per cent. stock to pay 154*l.* 10*s.* a year?

(4) I buy into the 3 per cents. at 92, hold for 5 years, and then sell at 84. What rate of interest have I realised?

(5) I buy into certain 4 per cents. at 75 and afterwards sell for 95. I have made $7\frac{1}{2}$ per cent. per annum. How long did I hold the stock?

(6) I bought into the 3 per cents. at 92. They fall to 85, and I am now offered a safe investment paying 5 per cent., but not subject to fluctuation of value. How long must I hold it before I shall make a profit by the change, in case the funds should rise to their former value?

4. (1) What is the compound interest on 1,875*l.* for four years at 5 per cent?

SOLUTION. At the end of each year the *amount* has increased in the ratio of $\frac{105}{100}$ or by the multiplier 1·05. As there are four years, we must use this multiplier four times.

$$1{\cdot}05 \times 1{\cdot}05 \times 1{\cdot}05 \times 1{\cdot}05 = 1{\cdot}21556 \text{ nearly}$$
$$1875 \times 1{\cdot}21556 = 2279{\cdot}176$$

	=2279*l.*	3	$6\frac{1}{4}$
Subtract .	1875	0	0
Remains, interest,	£404	3	$6\frac{1}{4}$

(2) Find the compound interest of 927*l.* for 3 years at $4\frac{1}{2}$ per cent.

(3) Find the compound interest of 3,750*l.* for 6 years at 10 per cent.

(4) Find the sum which, put out at compound interest at 10 per cent. for 6 years will amount to 3,750*l.*

(5) Find the present value of 2,279*l.* 3*s.* $6\frac{1}{4}$*d.* payable 4 years hence at 5 per cent.

(6) Find the present value of 10,000*l.* payable 4 years hence at 15 per cent.

5. (1) The population of a country has increased in the ten years 1861–71 from 2,943,578 to 3,408,766. What may it be expected to reach in 1891?

(2) The population of another country in the same time has diminished from 1,081,954 to 967,385. What may be expected to be its population in 1891?

(3) The population of an island is 35,743. There is no emigration or immigration. The annual deaths are 27 in the 1,000 and the births 42 in the 1,000. What will be the increase of the population in five years?

Profit, Estimates, and Partnership.

Very important applications of arithmetic fall under these heads. They are too varied to fall under any specific arithmetical rule, but a little common sense and attention will generally enable us to work them by a suitable application of the ordinary rules of arithmetic. For example, I buy a ton of coffee at 8*d.* per lb. How much must I sell it at to make 20*l.* profit?

Since the whole quantity is a ton, the profit on each pound of coffee will be $\frac{4800}{2240}$*d.* This added to the cost price will give the selling price, which is therefore $8\frac{15}{7}$*d.* = $10\frac{1}{7}$*d.*

Profit per cent. is an expression rather vaguely used. The percentage is sometimes taken on the selling price and sometimes on the cost price. Thus, formerly in the book trade it used to be said that the retailer's profit was 25 per cent., meaning that a retail bookseller gave 75*s.* for a book which he sold for 100*s.* But this profit was more than 33 per cent. on the outlay. Care must therefore be taken to express distinctly which is meant. The profit on a single transaction or set of transactions by no means represents a net

profit, as it is not charged with a variety of expenses which belong to the business in general rather than to the set of transactions in question. The following examples will illustrate these remarks.

In most manufacturing establishments of moderate size the cost of production divides itself into three heads :

1. Charges in direct proportion to the quantity produced, such as those for raw material, fuel, labour, &c.

2. Fixed annual charges, such as rent, interest of capital, and sometimes management.

3. Charges of an intermediate character, arising from its being cheaper or dearer to work, or buy and sell, on a large scale.

The question of profit and loss depends upon the joint effect of these three classes of causes. It is generally possible, for rough estimates—and such estimates are nothing more—to divide the third class between the first and second.

An example of the questions that arise out of this is as follows :

A maker of escape-wheels of a particular class gets 2*s.* 9*d.* each for them. The materials, fuel, gas, &c., used in making them come to 10½*d.* each, while his living, rent, &c., cost him 29*s.* a week. How many must he turn out weekly before he can save?

Here the net earning on each is 1*s.* 10½*d.*, and he must make as many as will bring these earnings to 29*s.*, that is to say—

$$\frac{29s.}{1s.\ 10\frac{1}{2}d.} = \frac{29 \times 24}{45} = \frac{464}{30} = 15\tfrac{1}{2} \text{ nearly.}$$

Therefore if he makes 16 a week, he will be able to save 11*d.*

The following example illustrates a more complicated case :

a. A firm turns out 50 tons of steel goods a week, using up for that purpose 51 tons of iron at 6*l.* 15*s.* per ton, 100 tons of coal at 11*s.* 6*d.*, and 45*l.* worth of other materials. Rent, rates, and taxes 219*l.* a year. Wages and other

expenses 75*l.* a week. How much per cwt. must the steel be sold for to give neither profit nor loss?

b. Assuming 10 per cent. of bad debts and a commission of 5 per cent. on all payments, what must be the selling price of the steel for the works to give 8 per cent. annual profit on a capital of 35,000*l.*?

The first thing is to reduce the rent, rates, and taxes to a weekly instead of an annual charge. 219*l.* a year is $\frac{219}{365}$ or $\frac{3}{5}$*l.* per day, which is $\frac{21}{5}$*l.* = 4*l.* 4*s.* a week. The first part of the question, then, stands as follows:

		£	*s.*	*d.*
51 tons iron at 6*l.* 15*s.*		344	5	0
100 tons coal at 11*s.* 6*d.*		57	10	0
Other materials		45	0	0
Rent, rates, and taxes		4	4	0
Wages and other expenses		75	0	0
Cost of producing 50 tons	10)	525	19	0
(The cost per cwt. is found by dividing this by 1000.)	10)	52	11	11
	10)	5	5	2
			10	6

And in order to give neither profit nor loss, this 10*s.* 6*d.* per cwt. must be the selling price of the steel.

For the second part of the question, the profit on capital may be regarded as part of the cost of production. It would be so, in fact, if the capital were borrowed at 8 per cent. interest. This profit is 2,800*l.* a year or $\frac{2800}{52}$*l.* = 53*l.* 16*s.* 11*d.*, making the cost of production for 50 tons

525*l.* 19*s.* + 53*l.* 16*s.* 11*d.* = 579*l.* 15*s.* 11*d.*

Next: 10 per cent. of bad debts means that 10 do not pay for 90 who do, that is, that 100*l.* must be charged to get 90*l.* The selling price must therefore be raised in the ratio $\frac{10}{9}$.

Again, the commission is paid on the money actually received; to provide for it, the money receivable must be raised in the ratio $\frac{20}{19}$. Hence the selling price must be raised altogether in the ratio $\frac{10}{9} \times \frac{20}{19} = \frac{200}{171}$. The selling price for the 50 tons will therefore be $\frac{200}{171}$ × 579*l.* 15*s.* 11*d.*, and the selling price per cwt. will be $\frac{579l.\ 15s.\ 11d.}{855}$ = 13*s.* 7*d.* nearly.

In taking the year as 52 weeks, we make it 364 days instead of 365. This inaccuracy would not affect the result perceptibly, and is moreover an 'error on the right side' in the estimate—that is, in favour of the producer.

In what proportion must coffee at 10*d.* and chicory at 5*d.* a lb. be mixed so as to leave a profit of 2*d.* a lb. when sold for $10\frac{1}{2}$*d.*?

The difference between the price of chicory and coffee is 5*d.* per lb. Now 1 lb. of coffee costs 10*d.*, and if we take away a part of the pound of coffee and replace it by the same part of a pound of chicory we shall reduce the cost price by *that part* of 5*d.* Again, the cost price has to be reduced by $1\frac{1}{2}$*d.* (inasmuch as the mixture is to cost $8\frac{1}{2}$*d.* a lb.), which is $\frac{3}{10}$ths of 5*d.* Therefore 1 lb. of the mixture must contain $\frac{3}{10}$ths of a pound of chicory.

A and B establish a business, A advancing 7,000*l.* and B 5,000*l.* for the purpose. Four years afterwards A advances 3,000*l.* more, and B 5,000*l.* more. At the end of 10 years from its establishment, C buys it for 34,000*l.* How shall this be divided between A and B?

The whole profit is 14,000*l.*, and this was obtained by the employment of 12,000*l.* for 10 years and of 8000 for 6 years. This is equivalent to the employment of 1000*l.* for 168 years. To each 1000*l.* for each year, therefore, is due a profit of $\frac{14000}{168}$*l.*

A has 7 thousands for 10 years and 3 for 6 years, or $7 \times 10 + 3 \times 6 = 88$; while B has 5 for 10 and 5 for 6 years, or $5 \times 10 + 5 \times 6 = 80$. Therefore A must take $\frac{88}{168} = \frac{11}{21}$, and B $\frac{10}{21}$ of 14,000*l.* Now $\frac{10}{21} \times 14{,}000 = \frac{20000}{3} =$ 6666*l.* 13*s.* 4*d.* So that this is B's share, while A's share is the remainder, or 7333*l.* 6*s.* 8*d.* of the profits, in addition to the repayment of 10,000*l.* to each.

In this case the profit is reckoned at simple interest, and is found to be at the annual rate of $\frac{1400}{168}$*l.* $= \frac{25}{3} =$ 8*l.* 6*s.* 8*d* per cent.

It would have been more strictly correct to treat the profits as compound interest. But this would have required the use of algebra, and even so only an approximate solution could have been obtained. Indirect questions of compound interest, especially when the rate of interest has to be determined, are quite beyond the reach of mere arithmetic.

I draw off 5 gallons from a 20-gallon cask of brandy, and

fill the cask up with water. I afterwards draw off 5 gallons again, and fill it up with water. What is the strength of the remainder?

I draw off each time one quarter of the contents, and consequently one quarter of the brandy which remains in the cask. At the second drawing, therefore, I take out one quarter of 15 gallons, and leave three quarters of 15, or $11\frac{1}{4}$ gallons. The strength, therefore, is as $11\frac{1}{4}$ gallons of brandy to $8\frac{3}{4}$ of water, or as 45 : 35, or as 9 : 7.

More shortly. After each drawing there remains three quarters of the brandy that there was before. Therefore, after the second drawing, there remains $\frac{3}{4} \times \frac{3}{4} = \frac{9}{16}$ of brandy.

The Chain-Rule.

This is a form of continued proportion used in the adjustment of exchanges and in the conversion of measures through intermediate steps of comparison. The simplest case is where a succession of units of measure are given, each in terms of the preceding one, and we want to find the last in terms of the first. It sometimes happens that in place of giving a unit of one measure in terms of the other, the relation is stated in the form, that so many of one are equivalent to so many of the other. This merely substitutes a vulgar fraction for a decimal in the multiplication.

For instance, the old Paris foot = ·32484 mètres, the mètre = 3·2809 English feet. The English foot = ·97114 Prussian foot. To express the Paris foot in Prussian feet we have only to multiply all these numbers together, which gives a Paris foot = ·32484 × 3·2809 × ·97114 = 1·035 Prussian foot.

The general case is of this character. Required the value of 3760 zwanzigers in pounds sterling, having given the following data as current rates of exchange:

617 zwanzigers	=	100 American dollars.
11 dollars	=	57 francs.
11 francs	=	3 reichsthalers.
69 reichsthalers	=	64 roubles.
32 roubles	=	5 sovereigns.

We have simply to multiply the given number of zwanzigers (3760) by the successive ratios by which we pass from each denominative to the next, that is to say,

$$\frac{100}{617} \times \frac{57}{11} \times \frac{3}{11} \times \frac{64}{69} \times \frac{5}{32} \times 3760$$

$$\text{Or } \frac{171 \times 3760000}{617 \times 121 \times 69} = \frac{642960000}{5151333} = 124{\cdot}8$$

The more usual arrangement, however, is as follows:—

617 zwanzigers	=	100 dollars.
11 dollars	=	57 francs
11 francs	=	3 reichsthalers.
69 reichsthalers	=	64 roubles.
32 roubles	=	5 sovereigns.
? sovereigns	=	3760 zwanzigers.

And the blank is to be filled up so that the product of each column shall be the same. We can evidently *divide out* common factors from each, and we get as before—

124·8 sovereigns = 3760 zwanzigers.

Examples.

1. A B and C enter into a speculation with a stock of 638*l.*, upon which they make a profit of 90*l.* A had 168*l.* in the concern 5 months, B had 70*l.* in for 8 months, and C 400*l.* for 7 months. How should the profit be shared?

2. A hires pasturage for 56 bullocks for 150 days for 5*l.*, but sells off 40 after 90 days, keeping the remainder. He takes in 18 bullocks belonging to B for 10 days, and again 5 bullocks for 12 days. He also lets C put in 36 bullocks for 60 days. What shall B and C pay?

3. A certain alloy consists of 19 parts by weight nickel, 17 lead, and 41 tin. The only nickel I can obtain is 10 lbs. of an alloy containing 11 parts of nickel to 7 parts tin and 5 lead. How much lead and tin must I add to make up the alloy I want?

4. I buy $2\frac{1}{2}$ tons of an alloy containing 7 parts by weight of nickel, 5 of tin, and 4 of lead, at 5*d.* a pound, and also

2 tons 7 cwt. of another alloy containing 17 nickel, 20 tin, and 10 lead, at $5\frac{1}{2}d.$ a pound. I melt them down with 3 cwt. of lead at $1\frac{1}{2}d.$ per lb. I sell the 5 tons, charging at the rate of 1*s.* a lb. for the nickel it contains, 5*d.* a lb. for the tin, and 2*d.* for the lead. What is my profit per cent. on the outlay?

5. A brickmaker pays 200*l.* a year rent and 7*s.* in the pound rates and taxes, as well as 2*s.* 6*d.* dues per 1000 of bricks. The other fixed charges are 250*l.* a year, and the cost of labour, coals, and so forth, is 12*s.* 6*d.* per 1000. How many thousand bricks a week must he turn out to make 20 per cent. on the cost price, and sell them at 30*s.* per 1000?

6. A steam vessel going 10 knots an hour consumes half a ton of coal per knot at 12*s.* a ton, and costs 10*l.* a day for wages, repairs, and interest of capital. She carries 3000 tons of coal and cargo. What is the greatest paying distance she can go, supposing she earns a farthing per ton freight per nautical mile?

7. A certain article is now imported duty free, and it is known that every penny per lb. of customs duty would diminish the consumption by one-tenth of the present importation. What rate of duty (to a whole farthing) would yield the greatest revenue?

8. The old wine gallon is 231 cubic inches. The cubic inch is ·000016386 cubic mètres, and the imperial gallon is 4·54102 litres. How many imperial gallons are there in 157 wine gallons?

9. The zoll-centner is 110·231 English pounds. The English pound is 453·59 grammes. The kilogramme is 2·442 Russian pounds. How many roubles per Russian pound is 375 reichsthalers per centner, reckoning a rouble at ·15625 of a pound sterling, and a pound sterling at 7 reichsthalers?

10. A certain estate in Russia is valued at 1750 roubles the dessiatine. A dessiatine is 10925 mètres, and an acre is ·40467 hectares. A pound sterling is 25·25 francs, and a

rouble is 3·95 francs. Express the valuation in *l. s. d.* per acre.

11. Express miles an hour in mètres per second, having given a mètre = 39·37 inches.

12. Taking the mean length of a minute of latitude as 6076·5 feet, and the French mètre as intended to be one ten-millionth of the distance from the pole to the equator, express the mètre in inches.

Specific Gravity.

This is, in point of fact, a ratio by means of which the relative heaviness of different substances is considered without reference to actual weight or bulk. This ratio is obtained by comparing the weights of similar bulks of two substances. For convenience it is usual, instead of comparing each substance in common use with every other, to compare all of them separately with one or two standard substances. We generally take water as the standard of weight for all solids and liquids, and dry atmospheric air with the barometer at 30 inches as the standard of weight for gases. The specific gravity, then, of a solid or liquid is the weight of any unit of bulk of that substance divided by the weight of the same unit of bulk of water.

For instance, the weight of a cubic foot of a particular specimen of iron is found to be 467 lbs. 6 oz.=7478 oz. Now the weight of a cubic foot of water is 997 oz. The specific gravity of iron is therefore said to be 7·5. The practical meaning of this is that any given bulk of this iron weighs 7½ times as much as an equal bulk of water. The specific gravity is not the most convenient form in which to record the weight of materials for the purposes of the engineer or contractor. The weight per cubic foot is better for his purpose. But the specific gravity has the advantage, being independent of the unit of bulk, of being the same for all countries and all languages, however the standards may vary. Moreover, as the specific gravity is generally obtained

by weighing an object against water, it is the most direct result from the experiment.

The specific gravity of liquids is generally obtained by means of a bottle which is ground so as to contain exactly 1000 grains (or other units of weight) of water; being carefully dried, it is then filled with the given liquid and weighed. The known weight of the bottle being subtracted gives the net weight of the liquid in grains, and we have only to mark off three figures of decimals to get the specific gravity of the liquid.

The specific gravity of a solid heavier than water and not soluble in it is found by weighing it, first dry and then in water. The difference between the two weights is evidently the weight of the water which has been displaced by the solid, that is to say, of an equal bulk of water. Hence the specific gravity will be got by dividing the dry weight by this difference. Thus if a lump of iron is found to weigh 15 oz. dry and 13 oz. in water, an equal bulk of water must weigh 2 oz., and the specific gravity of the iron will be $\frac{15}{2} = 7{\cdot}5$.

If, on the other hand, the substance will not sink in water, it must be loaded with some known weight of a substance whose specific gravity is also known, and that of the first substance must be inferred by calculation.

Thus, suppose that a pound of poplar wood is sunk by the help of two pounds of iron, specific gravity 7·2, and that the total weight in water is ·0556 lbs., we know that the bulk of the iron and wood together will be the same as that of (3 − ·0556 =) 2·9444 lbs. of water. But the bulk of the iron is that of $\frac{2}{7{\cdot}2}$ lbs. = ·2778 lbs. of water. The bulk of the pound of wood is consequently that of 2·9444 − ·2778 lbs. = 2·6666 lbs. of water. Its specific gravity is therefore $\frac{1}{2{\cdot}6666} = {\cdot}375$.

To obtain accuracy many precautions are necessary, such as varnishing the wood, and so on. In some cases it is more convenient to ascertain the bulk by actual measurement and then weighing. For the sake of simplicity I have neglected the weight of the air. The consequence is that the specific gravity thus obtained is incorrect, for the weight of the water and of the substance are each diminished by that

of an equal bulk of air. Thus, if a cubic foot of limestone weighs 172 lbs in air, and a cubic foot of water 62·365 in air, if we neglect the weight of the air, we get—

$$\text{Specific gravity} = \frac{172}{62 \cdot 365} = 2 \cdot 757957.$$

But a cubic foot of air weighs ·08 lbs. at 40° Fahrenheit and atmospheric pressure. The true specific gravity is therefore—

$$\text{Specific gravity} = \frac{172 + \cdot 08}{62 \cdot 365 + \cdot 08} = \frac{172 \cdot 08}{62 \cdot 445} = 2 \cdot 754424.$$

The difference of these is ·003533, or about $\frac{1}{1000}$ of the whole.

This shows very well what is the proportionate error made by neglecting the weight of the air. For substances very much lighter than water the error is greatest. For those about the same weight as water this error vanishes, and for very heavy substances the error is also small.

The chief problems relating to specific gravity are the following. Given the quantities and specific gravities of two or more substances separately to find the specific gravity of the mixture. If the quantities be given by bulk, or by cubic measure, it is sufficient to multiply each quantity by its specific gravity, add the products, and divide by the total quantity.

Thus, if we melt together 15 cub. in. of copper sp. gr. 8·7 with 5 cub. in. of tin sp. gr. 7·4, and 2 cub. in. of lead sp. gr. 11·4, the sp. gr. of the bronze will be—

$$\frac{15 \times 8 \cdot 7 + 5 \times 7 \cdot 4 + 2 \times 11 \cdot 4}{15 + 5 + 2} = \frac{190 \cdot 3}{22} = 8 \cdot 65.$$

But if the quantities be given by weight, we must take the sum of the weights for numerator, and the sum of the relative bulks, obtained by dividing each weight by its specific gravity, for denominator.

Thus, in the previous Example, if the quantities had been given in ounces instead of in cubic inches, we should have had—

$$\frac{15 + 5 + 2}{\frac{15}{8 \cdot 7} + \frac{5}{7 \cdot 4} + \frac{2}{11 \cdot 4}} = \frac{22}{2 \cdot 575} = 8 \cdot 544,$$

supposing that no dilatation or contraction took place. But if there is a contraction from the original bulk, say in the ratio of 1 : ·99, the specific gravity will be increased in the ratio $\frac{100}{99}$ and will therefore be 8·65.

The other practical question is: Having found the specific gravity of a mixture in unknown proportions of two substances whose specific gravities are separately known, to find the proportions in which they are mixed. This requires the solution of an algebraic equation, of which we only give the result here. Suppose, to fix our ideas, that we have a lump of gold quartz whose specific gravity we will call W, and we will call the specific gravities of gold and quartz G and Q respectively. The quantity of gold in an unit of weight of the mixture will be $\frac{G}{W} \times \frac{W-Q}{G-Q}$ and the quantity of quartz will be $\frac{Q}{W} \times \frac{G-W}{G-Q}$. This again supposes that there is no dilatation or contraction.

The temperature at which the specific gravity is recorded also requires to be noted for very exact purposes. But for ordinary work, provided there be no change of temperature during the experiment, the comparison of solids and liquids with one another, and of gases with gases, is not sensibly affected by this; for we may assume that the dilatation or contraction due to a small range of temperature affects them all equally. But when we compare gases and liquids, this is not the case, for the dilatation of liquids by heat is very small, so long as they remain liquids, while the dilatation of gases is very great.

For instance, the expansion of bulk from freezing to boiling point is for

Perfect gas	·365
Common air	·366
Pure water	·04775
Mercury	·01815
Iron	·0035

that is to say, gas expands from 1 to 1·365, water from 1 to 1·04775, and iron from 1 to 1·0035.

It is worth remark that in the metric system, the number of kilogrammes to the cubic décimètre (or litre) expresses the specific gravity referred to water as unity.

Examples.

(1) Find the weight per cubic foot of concrete built of granite (specific gravity 2·7), and cement (specific gravity 1·7), supposing the proportion of granite to be $\frac{2}{3}$ths of the whole in weight. Cubic foot of water = $62\frac{1}{2}$ lbs.

(2) What will be its specific gravity when soaked with water, supposing it to absorb $\frac{1}{3}$th of its weight without altering its bulk.

(3) Taking the specific gravities of gold and quartz as 19 and 2·6 respectively, find the quantity of gold per oz. of a mixture whose specific gravity is 7.

(4) Calculate its value, supposing the quartz to be worthless and the gold 3*l.* 17*s.* $10\frac{1}{2}$*d.* per oz.

(5) Find the bulk of the gold contained in a cubic inch of the mixture.

(6) A piece of glass whose specific gravity is 2·4, and whose weight is 2 lbs., is found to weigh 1 lb. when weighed in a certain liquid. What is the specific gravity of the liquid?

(7) A lump of iron weighing 1 lb. is found to require a pressure of 12·4 oz. avoirdupois to keep it immersed in mercury whose specific gravity is 13·5. What is the specific gravity of the iron?

(8) A gallon measure of dry sand weighs $16\frac{1}{4}$ lbs. As much water is then poured into it as it will absorb, and it is found to weigh 20 lbs. What is the specific gravity of the material of which the sand is composed?

(9) A crystal weighs 1·53 oz. in water and 1·73 in naphtha, specific gravity ·85. What are its actual weight, bulk, and specific gravity?

(10) 20 lbs. weight of an alloy whose specific gravity is 8·47, is melted with 15 lbs. of an alloy whose specific gravity is 11·15. The specific gravity of the mixture is found

to be 9·5. How much per cent. was the contraction or expansion?

Use of Tables.—Proportional Parts.

A table of single entry contains the result of a series of observations or of a series of arithmetical operations, more or less complicated, and arranged either by equal intervals or by a series of steadily increasing or decreasing numbers. The series by which the table is arranged is called the *Argument.* The result given in the table is called the *Tabular Entry*, or more simply the Entry. Such tables are generally arranged in double columns, the Argument in the first and the corresponding Entry on the same line in the second column. The following is a specimen taken from the middle of a Table of the Reciprocals of numbers:

Numbers	Reciprocals	Numbers	Reciprocals
51	·019607843	56	·017857143
52	·019230769	57	·017543860
53	·018867925	58	·017241379
54	·018518519	59	·016949153
55	·018181818	60	·016666667

If we want to find the reciprocal of any whole number between 50 and 60, we can take it out at once from this table. For instance, the reciprocal of 54 is ·018518519.

But it does not generally happen that we have to consult a table which gives us the exact entry which we require. For instance, it is quite likely that we might have required to consult that table in order to find the reciprocal of 54·4, instead of either 54 or 55. The question is how are we to make it serve this purpose. For this we must make the assumption that the tabular entry increases or decreases by uniform steps as the argument increases, also by uniform steps. This assumption is manifestly untrue, and therefore the result will not be exact; still it will be more nearly true than if we take any entry in the table as it stands. We shall revert by and by to the quantity of the error involved.

The arithmetical consequence of our assumption is this: That if we take a number half way between 54 and 55, the correct entry will be half way between the entries belonging to 54 and 55, viz. ·018518519 and ·018181818, and that this will be true or sufficiently near the truth, if for *half* we substitute any other fraction of the interval.

The first thing is to find what is half way between two numbers; it is only necessary to subtract one from the other and to add one-half of this difference to the first entry if the entries increase, and to subtract it if they decrease. The work of finding it in the above case is as follows:

```
54      ·018518519
55      ·018181818

      2)·000336701

        ·000168351 subtract from
        ·018518519

54·5    ·018350168
```

Now the reciprocal of 54·5 is ·01834862, and the result is not very accurate. The reason of the want of accuracy arises from the fact that in this particular example it is not *nearly true*, that the tabular entries diminish uniformly.

As far as the rule is concerned, it is the same for any intermediate fraction as for one half. Thus for 54·358 the work would have stood:

```
     54  ·018518519
     55  ·018181818

         000336701 multiply by ·358
               853

           1010103
            168351
             26936

           1205390 subtract from
        ·018518519

54·358  ·018397980
```

This of course will be inaccurate too, for the same reason, and nearly to the same extent. The true value is ·01839656.

This does not appear a very promising process; but there is this peculiarity about most tables, that if you make the table larger by subdividing the interval of the argument you may make it very nearly true, that the entries shall increase or decrease uniformly, and how near this is true may be found from the table itself by simply seeing whether the differences between successive entries vary materially or not; for if they do not vary at all it is exactly true, and then the rule holds perfectly. We will now show, in the case of a table of reciprocals, how rapidly this tends to become true as we subdivide the argument.

I will start with

Numbers	Reciprocals	Differences
54	·018518519	·000336701
55	·018181818	·000324675
56	·017857143	

Here the differences disagree after the fourth decimal place. I next take the part of the table where the interval is only one-tenth of a unit:

Numbers	Reciprocals	Differences
54	·01851852	·00003423
54·1	·01848429	·00003411
54·2	·01845018	

Here not only is the difference itself thrown back from the fourth to the fifth decimal place, but the differences agree much more nearly. In fact, the difference of the differences is ·000012026 in the first table, and ·00000012 in the second. Now this it is which measures the amount of the error. Thus dividing the interval by 10 has divided the error of the rule by 100.

Hence, in very large tables, this method of proportional parts may be used safely. In very small tables, or sometimes

near the beginning and end of large tables, it is only a rough approach to the truth.

In many tables it is usual to give a column containing the differences. This saves the subtraction. In logarithm tables this is done almost universally, and in some of the larger ones little tables of proportional parts are given—that is, the actual multiples of the difference by ·1, ·2, ·3, &c. up to ·9.

As an example, let us find the reciprocal of 54·05 from those of 54 and 54·1 given above. Multiplying 3423 by ·5 the product is 1711. Subtracting this from ·01851852
1711
we get, for the reciprocal of 54·05, ·01850141
By an independent calculation it is found to be ·01850139. This is pretty exact.

Examples of this method will be found in most books of logarithms. But it is worth while to remark that the rule is general.

For more exact rules of finding intermediate entries, a knowledge of mere arithmetic is not sufficient.

The rule may be used for finding the argument from the entry. For instance, to find the number whose reciprocal is ·01850141, I should have subtracted this from ·01851852 leaving ·00001711, and then I should have said the error on the entry is proportional to the error on the argument. This would have given me the rule of three sum—

3432 : 1711 :: ·1 : 0·05 very nearly.

And therefore I should have added ·05 to 54, making 54·05 for my answer.

As another example I will take the following from a table of square roots :

Numbers	Square roots	Differences
8390	91·5969431	54685
8391	91·6024916	54582
8392	91·6078598	

Required (1) the square root of 8391·427 and (2) the number whose square root is 91·6.

(1)	·427 × 54582 =	23306
	8391 . .	91·6024016
	Answer . .	91·6047322
(2)		91·6
	8390 .	91·5969431
	.	·0030569
	$\frac{30569}{54685}$ =	·55900
	.	8390
	Answer .	8390·55900

It is evident that these answers are not reliable beyond the fifth decimal place in (1) and the third place in (2). For the differences vary by 1 in the fifth place instead of being uniform or constant, and supposing the variation to be steady, the proportion by which we obtain the result cannot be exactly true.

Further explanations of this subject are given in most tables of logarithms and in trigonometry.

It frequently happens that there are tables in which the argument itself does not vary quite regularly. In this case it still remains approximately true that 'part way between the entries corresponds to part way between the arguments' —the part way being the same in each case.

Double entry tables are where the entry depends upon two variable things instead of one, so that we want not only one column with a lot of lines, but also a lot of columns. The multiplication table is a simple example of this. The multiplicand is one argument, the multiplier the other, and the product is the entry.

The rule of proportionate parts is the arithmetical equivalent of the geometrical one that a very short portion of a curve may be regarded as differing very little from a straight line.

The rule of *false position* depends upon the same principle. I do not think it advisable to give it here, because it requires some knowledge of algebra to know where it applies.

The tables in common use are chiefly the following :

(1) Logarithms of numbers.

(2) Trigonometrical logarithms.

(3) Trigonometrical tables, not logarithmic.

(4) Tables of squares, cubes, and square and cube roots.

(5) Tables of the reciprocals of numbers.

(6) Extended multiplication tables, including ready reckoners, and the comparison of measures.

(7) Tables of interest, annuities and leases.

(8) Tables of mortality, life assurance, and annuities for life.

Besides these, nearly everything that is occasionally wanted, and is difficult either to remember or calculate, has been published at some time or other in a tabular form. Each branch of science, and almost every trade, has its special tables. The best advice to be given is not to procure a table until it is absolutely or frequently needed, and then to procure as small a one as will answer the purpose. All ordinary work can be done with a small table of logarithms to five decimal places. Unless a man's work be very special, it is generally clumsy to use larger tables.

Number	Square root	Cube root	Reciprocal ·00
250	15·811	6·300	4000
251	15·843	6·308	3984
252	15·875	6·316	3968
253	15·906	6·325	3953
254	15·937	6·333	3937
255	15·969	6·341	3922
256	16·000	6·350	3906
257	16·031	6·358	3891
258	16·062	6·366	3876
259	16·093	6·374	3861
260	16·125	6·383	3846

Using the table given above—

(1) Find the square root, cube root, and reciprocal of 252·4.

(2) Find the number whose cube root is 6·37 and its square root and reciprocal.

(3) Find the number whose square root is 15·95, and its cube root and reciprocal.

(4) Find the square root and reciprocal of the number whose cube root is 6·329, without finding the number.

(5) Find the square root and cube root of the number whose reciprocal is ·003977.

(6) Find the number whose reciprocal is 39·14967, and state how many figures of your answer you consider reliable.

The student may set a great many similar examples for himself out of the little table here given. He may also consult the chapter on logarithms in the ALGEBRA and TRIGONOMETRY of this Series.

CHAPTER XV.

SQUARE AND CUBE ROOT.

Square and Square Root.

WHEN a number is multiplied by itself it is said to be squared, and the product is called the square of the number. Conversely, the number itself is called the square root of that product. Thus 7 × 7 = 49. Hence 49 is the square of 7, and 7 is the square root of 49. Similarly, 1024 is the square of 32, and 32 is the square root of 1024.

All numbers are not square numbers, that is to say, all numbers have not an exact square root. For instance, the square of 6 is 36, and the square of 7 is 49. Hence the numbers between 36 and 49 are not exact squares, for if they were, their square roots would be whole numbers lying

between 6 and 7, which is absurd. The following are the squares of the numbers up to 20. These should be committed to memory.

Numbers	1	2	3	4	5	6	7	8	9	10
Squares	1	4	9	16	25	36	49	64	81	100
Numbers	11	12	13	14	15	16	17	18	19	20
Squares	121	144	169	196	225	256	289	324	361	400

The square roots of numbers which are not square can only be expressed approximately. No fraction multiplied by itself will give a whole number for the product, and therefore no finite fraction can be the square root of a whole number. Accordingly, when we express an approximate square root by a decimal fraction, the decimals neither terminate nor circulate.

Squaring a number, whole or fractional, presents no difficulty, being mere multiplication, but it is worth while to pay some attention to the way in which increasing a number affects its square. Suppose I take the number 40, of which I know that the square is 1600, and suppose I also take another number 9, of which the square is 81, am I justified in inferring that the square of 49 is 1681? I say, certainly not; as a matter of fact $49 \times 49 = 2401$. We may perform the multiplication as follows, without actually working out the multiplication indicated, and the process will be instructive. We have to multiply $40 + 9$ by itself. Let us first multiply by 40 and then by 9, and add the products. Multiplying by 40 we get $40 \times 40 + 40 \times 9$. The multiplication by 9 gives $40 \times 9 + 9 \times 9$. The square of $40 + 9$ is therefore $40 \times 40 + 2 (40 \times 9) + 9 \times 9$. We thus see that the square of the sum of two numbers consists of three separate parts: 1st. The square of one number; 2nd. Twice the product of the numbers; 3rd. The square of the other number. This is evidently true, not only of the numbers 40 and 9, which we have taken as an example, but also of any num-

bers whatever. Generally, then, we may say that the square of $(a + b)$ is

$$a \times a + 2 \times a \times b + b \times b, \text{ or } a^2 + 2ab + b^2,$$

as algebraists write it.

This form of a square number should be thoroughly understood and remembered.

The square of a number consisting of several figures generally consists of twice as many figures as the number, unless the left-hand figure is 3 or less than 3. If it be 3, the number of figures in the square *may* be 1 less than double. If less than 3 it *must* be so.

If we square any number with noughts on the right hand we shall have twice as many noughts in the square; thus the square of 32 is 1024, and the square of 32000 is 1024000000. Hence a number with an odd number of noughts, as 10 or 1000, cannot have an exact square root. In the operation of extracting the square root of a whole number consisting of several figures, the first step is to mark off the figures into pairs, beginning from the right-hand side. This is generally done by putting a dot over the unit figure, the hundreds, the tens of thousands, the millions, &c. The next step is to find the highest square in the figure, or two figures to the left. Take, for instance, the case of 1857.

The greatest square in 18 is 16, which is the square of 4. Hence we know that the tens figure will be a 4, and we thus break up the number 1857 into $40 \times 40 + 257$. Now remember that the square of $a + b$ is $a \times a + 2 \times a \times b + b \times b$. Hence this remainder 257 will consist of $2 \times 40 \times b + b \times b$, where b represents the units figure. Now it must be remembered that $b \times b$ will be less than 100, and, at any rate, considerably less than $2 \times 40 \times b$. In order to find b by trial therefore, we neglect for the moment the term $b \times b$, and we consider this remainder 257 as if it were made up simply as $2 \times 40 \times b$. Then we find b by dividing by 80. This gives us 3 as a trial value of b; (but only a *trial* value), and we have then to see whether on finding the product 80×3, there still remains 3×3. In point of fact we find it to be so, and that there is a further remainder, after subtracting 249, of 8, and therefore that 1849 (the square of 43) is the greatest square number less

than 1857. Now suppose that we had two more figures, and that the number of which we wanted the square root was 185761 What we have done shows us that, neglecting units, the greatest square in 185761 is that of 430, namely, 184900 This leaves a remainder of 861, which, as before, must consist of 2 × 430 × *b* + *b* × *b* Getting the trial figure by dividing the remainder by 860, we find that *b* is 1 Now, making up the quantity 2 × 430 × *b* + *b* × *b*, by writing 1 for *b*, we find that it comes exactly to 861 There is then no further remainder and therefore 431 is the exact square root of 185761. This process is condensed as follows, all superfluous noughts being omitted

```
      18576̇1̇(431
       16
       ——
   83) 257
       249
  861)  861
        861
```

First, mark the figures into pairs, beginning at the right Then, taking the left hand group, which is here 18, I find the greatest square in it is 16, which I put down under it, and I put the corresponding figure 4 for the first figure of the square root I then subtract the 16 and take down the next two figures, 57 I have now to form my trial divisor Taking for the moment that the 257 represents units, the 4 which I have put down will be 40 I shall therefore have to divide the 257 by 80. I do not put down the nought, because I should have to strike it out again if I did On dividing by 80 I find the trial figure is 3 Now what I have to get, in order to make up the square so far, is 2 × 40 × 3 + 3 × 3 This is the same as 3 × 83 I therefore put down the 3 to the right of the 8 and that is my complete divisor I now simply do this step of division I get remainder 8 and I take down two more figures. I put down the 3, the second figure of the square root My next trial divisor will be 2 × 430 or 860, of which again I do not put down the 0, and I remark that instead of going back to the square root and doubling the 43, I might get the same result by simply doubling the 3 of the 83 which went before I find the trial divisor to be 1 This gives me the complete divisor 861, which 'goes once' without remainder The complete square root is therefore 431

It sometimes happens that the trial figure first found is too large, as in ordinary division. In that case, of course, smaller figures must be tried. The following is another example, worked at full

```
         670861801 ( 25901
         4
   [1] 45 ) 270
            225
     509 ) 4586
           4581
[2] 51801 )   51801
              51801
```

Decimals present no difficulty whatever. We have simply to point from the unit figure. If, for instance, the last number had been 67086·1801, it is clear that its square root would be 259·01, for we might have got rid of the decimals altogether by multiplying by the square of 100, and it is clear that, so far as the figures written down are concerned, the work would have been exactly the same as before. But it would have been a very different matter if we had had three decimals instead of four, for the pointing then would have been quite different. The work would now run—

```
            670861 801 ( 819·061
            64
      161 ) 308
            161
     1629 ) 14761
            14661
   163806 ) 1008010
             982836
  1638121 ) 2517400
            1638121
             879279
```

[1] 4 into 27 goes 6, but 46 × 6 gives 276, which is too large. We therefore take 5.

[2] On taking down 18, I find it will not go, because the real divisor is 5180 + something or other. I therefore take down two more figures, and put a nought in the square root.

This difference of result depends on the fact that the square root of 100 is 10, but that the square root of 10 is the interminable fraction 3·16227766017, of which we have put down only the first eleven decimal figures. As far as figures are concerned, it will be found that 819061 can be got from 25901 by either multiplying or dividing by the square root of 10.

Contraction of the Process.

Looking back to the last example it will be seen that the figure 1 in the last divisor, and any other figures which we may add to this last divisor, will not affect the three or four next figures, at least, of the square root. Hence we may get those figures by simply working on with simple division. Of course this division will be contracted in the ordinary manner. We will accordingly work the example for some few more figures in this manner, taking it up from the last divisor :—

```
1638121 ) 2517400 ( 153676
          1638121
          -------
           879279
           819065
           ------
            60214
            49144
            -----
            11070
             9829
             ----
             1241
             1147
             ----
               96
               98
```

The square root is therefore 819·06153676. We may in general work as many figures by division as we have already got by the exact process, the last figure so obtained being,

however, scarcely reliable. The rule for finding the decimal point is very simple. If the number contains whole numbers as well as decimals, put the decimal point in the root when you come to it in the number. But if you have to take the square root of a number like the following ·000003714, it is simply necessary to remember that each pair of figures in the number gives rise to a single figure in the root. Hence the root will begin with ·001.

Vulgar Fractions.

The square of a vulgar fraction presents no difficulty. It is simply the square of the numerator divided by the square of the denominator. It is to be remarked that if a fraction be in its lowest terms, its square will also be in its lowest terms. The square root of a fraction is found by taking the square root of both numerator and denominator; but this is of no practical use unless the numerator and denominator happen to be square numbers. It is generally best to begin by reducing it to its lowest terms, because otherwise there may be some factor in each which is not square and which therefore may introduce unnecessary decimals into each. Thus $\frac{242}{50}$ reduces to $\frac{121}{25}$, of which the square root is seen to be $\frac{11}{5}$, while if we took the square root as it stood, we should have $\frac{15{\cdot}5563492}{7{\cdot}0710678}$. Of course division would give us 2·1, which is the same result as before, but it would be obtained by a more roundabout and less accurate process.

When a fraction is not an exact square, the best way generally is to reduce it to a decimal fraction at once, and then to take the square root. There is sometimes an advantage in introducing a factor which will make the denominator a complete square. For instance, the square root of $\frac{47}{12}$ may be found either by reducing it at once to a decimal or by multiplying top and bottom by 3, which gives us $\frac{141}{36}$. We can then find the square root of 141, and divide that by

6. This is especially convenient when a table of square roots can be used.

It is worth while to remark that if a number contains any factor whatever, its square will contain the square of that factor.

The extraction of the square root answers the two following questions: 1. Let there be three quantities such that the first : second :: second : third, where the first and third are given. What is the second? Evidently the result of the proportion is that the product of the second into itself is equal to the product of the first and third, or, in other words, the second is the square root of this latter product. This is called finding a mean proportional. Thus to find a mean proportional between 7 and 63 we take the square root of 7×63, or of 441, viz. 21. This evidently satisfies the proportion, 7 : 21 :: 21 : 63.

2. Geometrically it solves the following question. Given the length and breadth of a rectangular figure, to find the side of a square which will have the same surface. This is merely the geometrical statement of the first question.

The sign of the square root is $\sqrt{}$: thus $\sqrt{9}=3$, $\sqrt{1369}=37$.

Cube and Cube Root.

The cube of a number is the product of a number and of its square, or the product of a number multiplied twice into itself. Thus the cube of 3 is $3 \times 3 \times 3 = 27$. The cube of 7 is $7 \times 7 \times 7 = 343$. Taking the cube is the direct answer to such questions as the following: A linear foot being 12 linear inches how many cubic inches is a cubic foot? The answer is the cube of 12, or $12 \times 12 \times 12$ or 1728. The process of taking the cube is merely one of multiplication, and calls for no special remark. The cube root answers the question: What is that number, which, multiplied twice by itself, makes up the number of which it is called the cube root? It answers the geometrical question: If a cube contains *so many* cubic units, how many linear

units will its side measure? The cubes of the numbers up to 12 should be committed to memory. They are as follows :—

Numbers	1	2	3	4	5	6	7	8	9	10	11	12
Cubes	1	8	27	64	125	216	343	512	729	1000	1331	1728

Since 1000 is the cube of 10, the cube root of 1000 times any number will have the same figures as the cube root of the number itself. Thus the cube of 62·3 is 241804·367. Hence the cube of 6·23 is 241·804367. But if we multiply a number by 10, its cube root will have a different set of figures and if we multiply the number by 100, we shall again have a different set of figures: thus

The cube root of	47	is	3·6088261
,, ,,	470	,,	7·7749801
,, ,,	4700	,,	16·7506869

Every number, therefore, considered with reference to its figures only, and not with reference to its decimal place, has 3 cube roots which differ from one another by the cube root of 10 and the cube root of 100. The extraction of the cube root is a troublesome process, not unlike that of extracting the square root, only much longer and more complicated. I have never known it used except as an exercise of mixed arithmetic and algebra, and therefore I do not give it here. The following process is much simpler. Suppose it be wanted to find the cube root of any number N. In the first place we find some number x whose cube X $(= x \times x \times x)$ is somewhere near the given number. Then the fraction, $\frac{2N + X}{N + 2X} \times x$, will be a nearer approximation to the cube root than x itself was. When we have found this value, we can take this as x and repeat the process. Thus, to find the cube root of 241·804367, I observe that 216, the cube of 6, is the nearest to 241. Hence the first value of x is 6 and X is 216. Therefore

$$\frac{2N + X}{N + 2X} \times x = \frac{699{\cdot}608734}{673{\cdot}804367} \times 6$$

$$= \frac{4197{\cdot}636404}{673{\cdot}804367} = 6{\cdot}23 \text{ very nearly.}$$

On trying $6{\cdot}23$ we find it is exact.

Let us take for another example the cube root of 47. The nearest cube to it is that of 4, namely 64. Here

$$\frac{2N + X}{N + 2X} \times x \text{ is } \frac{94 + 64}{47 + 128} \times 4 = \frac{4 \times 158}{175} = \frac{632}{175}$$

$$= \frac{2528}{700} = 3{\cdot}61 \text{ nearly.}$$

Now take $3{\cdot}61$ for x, and therefore

$X = 47{\cdot}045881$ and $\frac{2N + X}{N + 2X} \times x$ is $\frac{141{\cdot}045881}{141{\cdot}091762} \times 3{\cdot}61$

$= 3{\cdot}6088261$, which is exact to seven places of decimals.

We will take another example, not quite so favourable—the cube root of 19 . and we will begin by making $x = 2$, and therefore $X = 8$. We then have $\frac{38 + 8}{16 + 19} \times 2 = \frac{92}{35}$

$= \frac{184}{70} = 2{\cdot}63$ nearly. Now take this for x, then X

$= 18{\cdot}191447$ and $\frac{2N + X}{N + 2X} \times x = \frac{56{\cdot}191447}{55{\cdot}382894} \times 2{\cdot}63$

$$= 2{\cdot}6684 \text{ nearly.}$$

Next, making $x = 2{\cdot}6684$ we find $X = 18{\cdot}99996478$, and

we get $\frac{2N + X}{N + 2X} \times x = \frac{56{\cdot}99996478}{56{\cdot}99992956} \times 2{\cdot}6684$

$$= 1 - \frac{{\cdot}00003522}{56{\cdot}99992956} \times 2{\cdot}6684$$

$$= 1{\cdot}0000006179 \times 2{\cdot}6684$$

$$= 2{\cdot}6684016488$$

By Barlow's Tables I find it to be $2{\cdot}6684016$. It is therefore right so far at least. Assuming that there is no actual error of work, it is no doubt correct to the last figure but one.

The ordinary process of extracting the cube root is much more laborious for the same number of figures. Moreover

this method has the practical advantage that an error of work gets corrected at the next trial. Thus if I had written $x = 2{\cdot}6673$ instead of $2{\cdot}6684$, I should only have been a little farther off at the next step—say correct to 7 figures instead of to 9, perhaps.

The cube root of 10 is 2·1544347.

The cube root of 100 is 4·6415888.

Fourth Power and Root.

When a number is multiplied three times into itself, it is said to be raised to the fourth power; that is, if $A = a \times a \times a \times a$, A is said to be the fourth power of a, and a is said to be the fourth root of A. Practically, the fourth power may be considered as the square of the square, and similarly the fourth root may be considered as the square root of the square root. That is the best way of treating it arithmetically. But the fourth root may be obtained directly by a rule nearly similar to that given for the cube root, viz.—Let N be a given number, x a number whose fourth power $X = x \times x \times x \times x$ is not very different from N, then a nearer approximation to the fourth root will be $\frac{5N + 3X}{3N + 5X} \times x$.[1]

Examples.

1. Find the square roots of

(1) 4356	(2) 19044
(3) 43264	(4) 654481

[1] The general formula for the nth root is—

$$\frac{(n+1)N + (n-1)x^n}{(n-1)N + (n+1)x^n} \times x.$$

x being a first approximation. In the case of the square root this becomes $\frac{3N + x^2}{N + 3x^2} \times x$, which can be used as a check on the extraction of the square root, but can hardly be said to give shorter work. Making $n = 3$, we get the formula already stated for the cube root.

(5) 9345249 (6) 12299049
(7) 93470224 (8) 99980001
(9) 3323293056960I
(10) 9043820750088044900I

2. Take the square root of 1853020188851841, then take the square root of that square root, and so on, until it can no longer be done without decimals.

3. Find the square roots, to 7 decimal places, of

(1) 97199881	(5) 9719·9881	(9) ·97199881
(2) 9719988·1	(6) 971·99881	(10) ·097199881
(3) 971998·81	(7) 97·199881	(11) ·0097199881
(4) 97199·881	(8) 9·7199881	(12) ·00097199881

4. Find the square roots of the following numbers to 7 decimal places.

(1) 3877 (2) 3·877
(3) 5683 (4) 3·14159265
(5) ·03141593 (6) 6·28318531

5. A square plot of ground contains 320 acres. How many feet are there in a side?

6. A square plot of ground contains 31 sq. m. 365 a. or. 4 p. How many feet are there in a side?

7. What will be the size of a square bar weighing the same per foot run as a bar $2\frac{1}{2}$ in. wide and $\frac{5}{8}$ in. thick?

8. Gold weighs 1200 lbs. to the cubic foot and iron 480. What will be the size of a square bar of gold equal in length and weight to an iron bar an inch square?

9. Find a mean proportional between 162 and 1058.

10. Find a mean proportional between 3·3166248 and 23·2163735.

11. Find the cube roots of the following exact cubes·

(1) 592704 (2) 28652616
(3) 286191179 (4) 2869341461
(5) 344324701729 (6) 738518126319

12. Extract the cube roots of the following numbers to 6 decimal places:

(1) 2 (2) ·3 (3) 879 (4) 8·79 (5) 10000 (6) 100000

13. A perfectly squared brick wall is 2 feet thick, 12 feet high, and 1029 yards long. Supposing the bricks to have been built up in a cube, what would the side have been?

14. A bar of metal 5·75 feet long, 3·9 in. wide, and ·7 in. thick, is cast into a cube. What is the side of the cube?

15. The specific gravity of copper being taken as 8·75, what is the side of a cube of copper equal in weight to a cubic foot of water? And what is the side of a cubic vessel of water containing a quantity of water equal in weight to a cubic foot of copper?

CHAPTER XVI.

THE APPLICATION OF ARITHMETIC TO MACHINES, WORK, AND MOTION.

By a machine is to be understood any contrivance for converting force or continued pressure into mechanical work, for transferring mechanical work from one place to another, or for transforming it. A machine does not create work, but only transfers or transforms the mechanical work supplied to it. The meaning of the term *mechanical work* will be explained further on.

In this chapter I shall assume certain mechanical principles to be understood, and I shall state certain formulæ without proof. These are all elementary formulæ which may be found with their proofs in all good books on mechanics—a subject which I do not here profess to teach. My object in this chapter is to bring together certain appli-

cations of arithmetic which all mechanists need, but which, so far as I know, are only to be found scattered in treatises on mechanics, hydrostatics, and hydraulics. Incidentally, I give the arithmetic of falling bodies.

Mechanical Work and Power.

A force which has been allowed to act freely on a body, so as to move it from rest through a given space, is said to have done a certain quantity of mechanical work. In consequence a certain velocity will have been impressed on the moving body. To take this velocity out of it, and to bring it again to rest, requires an equal amount of work to be done in the opposite direction. In the same way any change of velocity requires an expenditure of mechanical work.

Mechanical work is measured by reference to the work done by a constant force. Such a force, for all ordinary practical purposes, is the force of gravity acting at or near the surface of the earth. The actual measure is the quantity of work done by gravity when a unit of weight is allowed to fall freely from rest through a unit of space. The more ordinary way of stating this is by an inversion; namely, by neutralising the work of gravity by lifting a body against it. The work done in lifting a body through a space against gravity, and leaving it at rest at the top of the lift, is the same as would be done by gravity in making it fall through the height lifted. The English measure of work is one pound lifted one foot high, and is called a FOOT-POUND. The French measure of work is one kilogramme lifted one mètre high, and is called a KILOGRAMMÈTRE.

Three pounds lifted one foot high is three foot-pounds, and one pound lifted three feet high is also three foot-pounds. Hence four pounds lifted seven feet high, or fourteen pounds lifted two feet high, is 28 foot-pounds. This facility of comparison is of great advantage in the application of the principle of mechanical work.

Thus if 1,000 gallons of water have to be raised to the height of 100 feet, the work to be done is (100 × 1000 × 10) a million foot-pounds, and you cannot get the water up unless you expend that amount of useful work upon it. You may waste part of your work, and you may spill part of your water, but if you have only half a million of foot-pounds at your disposal, you can no more do the work without adding more foot-pounds of work, than you can raise 1,200 gallons without adding 200 to the 1,000 gallons of water which you have in your tank.

The measure of work may be expressed not only in foot-pounds but in the product of any lineal measure into any weight; thus foot-tons, inch-pounds, grammètres. In England we use foot-pounds for ordinary work, foot-tons for heavy work, and inch-pounds for small work.

It is necessary to be careful not to confuse foot-tons with feet per ton. Thus one foot-ton, or ton-foot as it is sometimes called, is 12 × 2240 inch-pounds, while one foot per ton is $\frac{12}{2240}$ inches per lb., and one ton per foot is $\frac{2240}{12}$ lbs. per inch.

It is well to draw attention to the specific difference between work and pressure. A resisted pressure is not work. A weight resting on a table is not doing work. Work implies pressure acting through space, that is to say, moving something through some definite distance. It is pressure with motion, not pressure without motion.

A man supporting a weight is not doing external mechanical work. Whatever internal work may be going on in his muscles, the weight remains at rest, and the man's want of ease, or fatigue, is not external work.

Work may be expended in various ways besides lifting, as in friction, in the production of heat, in breaking up materials, in condensing or expanding gases. In speaking of the efficiency of machines for the transmission of work, all

these expenditures are called '*lost work.*' What remains is called the *useful work.* Thus if a steam engine exerts a million foot-pounds on a pump, and has only raised 500 gallons 100 feet, there is half a million foot-pounds of work as effectually lost as if the whole 1000 gallons had been raised and half of it 'spilt.' The ratio which the useful work got out of a machine bears to the work put into it, is called the efficiency of the machine. Thus, in the example just given the efficiency of the pumping gear, as a machine, is ·5.

In this illustration the water is supposed to be delivered from the machine and left at rest. If it was taken from rest, but retains velocity on its delivery, there is 'work left in it,' and that work must be taken out of it to bring it to rest.

Heat is now recognised as a particular form of motion. Variations of temperature involve changes of motion, and thereby involve work. In the steam engine the combustion of coal produces heat, and part of that heat is converted into mechanical work. A very small part only, however, is thus utilised. One pound of water, having its temperature lowered one degree of Fahrenheit's thermometer, gives out 772 foot-pounds of work. On the other hand, most applications of mechanical power expend part of their work in the production of heat, which is for the most part wasted.

For details of the conversion of heat into work the student must be referred to works on the steam engine or treatises on physics. Our concern is with the arithmetic of it. It may be stated, however, that 1 lb. of coal can under the most favourable circumstances be made to evaporate from 12 to 16 lbs. of boiling water, the evaporation of each pound being equivalent to 745,800 foot-pounds of mechanical work. At this rate 1 lb. of coal ought to give out from nine to twelve million foot-pounds of work, while, in reality, no steam engine ever does so much as two million foot-pounds for a pound of coal, so great is the loss from the want of

proper means of utilising the whole work of the combustion of the coal.

Power means the work which an engine can do in a given time. Thus a horse-power (English) is the capability of doing 33,000 foot-pounds of work per minute, or 550 per second. A French horse-power is 75 kilogrammètres per second, and therefore $\frac{75}{550} \times 7 \cdot 23314 = \cdot 9863$ British horse-power, or 542·5 foot-pounds per second.

There is a distinction to be drawn between three kinds of horse-power, all of which are quoted :

1. N. H. P. = Nominal horse-power.
2. I. H. P. = Indicated horse-power.
3. Effective horse-power.

The first is merely an arbitrary way of stating (for the purchase and sale of engines) the horse-power derived from a mode of measurement which is not applicable generally to modern steam engines. The second is found by ascertaining by means of an indicator the mean pressure per square inch on the piston, and then multiplying this by the area and travel. Apart from certain errors, it therefore really discovers the work done by the steam upon the piston in a given time. Still this is not *effective* power. To get that we must subtract all the work lost in the engine itself by friction, feed-pumps, and so on, or what is more easily stated, we must measure the work effectively applied by the engine to the driving shaft or pump-lever which it has to work. There is then generally a further loss of work in the machine which is driven by the steam engine, whether pump, paddle-wheel, screw-propeller, or winding-gear.

The arithmetical questions usually arising are such as these.

An engine is pumping water from a depth of 200 feet. The I. H. P. is 1375. The efficiency of the engine is ·8 and of the pump ·85. What will be the delivery per second?

$$\frac{\cdot 8 \times \cdot 85 \times 1375 \times 550 \text{ ft.-lbs.}}{200 \text{ ft.}} = 2571 \cdot 25 \text{ lbs.},$$

or 257 gallons per second.

One engine (A) is pumping 10,000 gallons of water per minute from a mean depth of 200 fathoms, on a consumption of 14,400 lbs. of coal per day, while another (B) is pumping 16,000 gallons from 60 fathoms on half the daily quantity of coal. Which is working most economically?

A does 12,000,000 ft.-lbs. per minute, while B does 5,760,000. For each pound of coal therefore A does 1,200,000 ft. lbs., and B does 1,152,000. It follows that A is working more economically than B in the ratio of 1,200 to 1,152, or of 25 to 24.

The mean indicated pressure in a steam cylinder is 35 lbs. per square inch, the piston area is 1,257 square inches, and the stroke 3 feet, the engine working 40 strokes per minute. What is the I. H. P.?

Each stroke involves a travel of 6 feet. The travel per minute (or mean speed of piston) is therefore 240 feet, and the I. H. P. is

$$\frac{1257 \times 240 \times 35}{33000} = 320.$$

It is not to be supposed that there is any minute accuracy in all this. Gravity is not absolutely a constant force as the height changes, and it varies in amount at different places. But these differences are far too small to be noticeable in common practice.

Speed or Velocity, and Time.

When a constant force acts upon a body at rest but free to move, the speed which it attains is proportional to the time, while the work is proportional to the space. Thus, if a weight of one pound be allowed to fall freely, we have, at the *end* of the number of seconds stated in the first column:

Time in seconds	Speed in ft. per second	Space in feet.	Work in ft.-lbs.
1	32·2	16·1	16·1
2	64·4	64·4	64·4
3	96·6	144·9	144·9
4	128·8	257·6	257·6
5	161	402·5	402·5

The relation between the columns is that the second column is simply the product of the first by 32·2. The third column is *half the square* of the first multiplied by the same number 32·2. The fourth column is the third multiplied by the weight. The third column may be obtained from the second by squaring and dividing by $2 \times 32{\cdot}2 = 64{\cdot}4$. In like manner the second column may be obtained from the third by multiplying by 64·4 and taking the square root.

I state the rule in this form because it is thus the general rule for finding the height due to a given velocity, or the velocity due to a given height. 32·2 is the multiplier for gravity when the time is given in seconds, and the measure of length in English feet. When the length is given in mètres, the multiplier is 9·815. Calling either of these g, to express the force of gravity by a single letter, we have—

$$\text{Velocity, } v = g\,t = \text{sq. root of } 2\,s\,g$$

$$\text{Time . . } t = \frac{v}{g} = \text{sq. root of } \frac{2\,s}{g}$$

$$\text{Height . } s = \tfrac{1}{2}\,g \times \text{square of } t.$$

$$= \tfrac{1}{2}\,v\,t.$$

$$= \frac{\text{square of } v}{2\,g}.$$

By the velocity due to a given height is meant the final velocity which a body acquires in falling freely from that height. The height due to the velocity is the converse, namely, the height needed to give the velocity. The body is supposed to fall freely, without resistance from the air.

When we do not start from rest, or stop at rest, but only alter the velocity, we must take the difference of velocity, and find the height due to that quantity. The work will thus be proportional to the height due to the difference of velocity. This is very easily seen in the case of gravity. For the work done in each second must be the same, and it will be seen that the difference of velocity in each second is the same, namely, 32·2 feet per second.

I cannot discuss here the calculations necessary when the force varies either in intensity or direction, or where the direction of motion is different at the inlet and outlet. These questions are better treated under mechanics or the geometry of motion, than under arithmetic. But the foregoing considerations give a direct answer to all questions of unresisted vertical motion, whether up or down. For instance:

A stone takes 9 seconds to reach the bottom of a shaft. What is the depth and with what velocity does it strike the bottom?

Answer: Depth $= 16·1 \times 9 \times 9 = 1,304$ feet

$v = 32·2 \times 9 = 289·8$ feet per second,

neglecting atmospheric resistance.

How long does a stone take to fall 1,000 mètres, and what velocity does it acquire?

Answer: Square of $t = \dfrac{2,000}{9·815} = 203·77$

$t = 14·27$ seconds

Square of $v = 2,000 \times 9·815 = 19630$

$v = 140·1$ mètres per second,

neglecting atmospheric resistance.

The mean velocity of the water issuing from a fountain-jet is 70 feet a second. The height of the jet averages only 57 feet. How much of the work is expended in overcoming the resistance of the air?

The height due to the velocity is $\frac{4{,}900}{64{\cdot}4} = 76$ feet, and the work stored in the issuing jet is proportional to this. The proportionate loss of work is therefore—

$$\frac{76 - 57}{76} = \frac{19}{76}$$

Water enters at the bottom of a vertical pipe with a mean velocity of 65 feet a second, and is driven out through an orifice half the size of the entrance, 80 feet above. What is the pressure per square inch at the entrance, neglecting the resistance of the pipe?

Since the same quantity must leave the pipe as enters it, the issuing velocity is 130 feet a second. The height due to this velocity is $\frac{16{,}900}{64{\cdot}4} = 262{\cdot}4$. Adding to this the 80 feet of the pipe, we have the pressure of a column of water 342·4 feet high, or 148 lbs. to the inch. Besides this there is the atmospheric pressure of 15 lbs. both inside and outside the pipe. But this, being balanced, does not affect the bursting pressure.

In this example I have used the mechanical principle that the velocity caused by the pressure of a certain head of liquid is the velocity due to a height equal to the head. It should also be noticed that the friction and throttling of the pipe would in practice form considerable sources of resistance.

An undershot water-wheel has a fall of 5 feet. The mean velocity in the head-race is 10 feet per second, and in the tail-race 4 feet. The area of the head race is 2 square feet, and the efficiency of the wheel and machinery is ·45. What will be the power?

The difference of velocity is 6 feet per second. The height due to this is $\frac{36}{64{\cdot}4} = {\cdot}56$. Adding this to the 5 feet fall, we have a total height of 5·56 ft. The quantity of water passing through is 20 feet per second, or 1,250 lbs. per second. The work per second is therefore $1{,}250 \times 5{\cdot}56$ ft.-lbs., and the horse-power taken out of the water is $\frac{1{,}250 \times 5{\cdot}56}{550} = 12\frac{7}{11} = 12{\cdot}64$. The effective horse-power is $12{\cdot}64 \times 0{\cdot}45 = 5{\cdot}7$.

Leverage.

Besides the foot-pound of work, namely, the work of raising one pound one foot against gravity, there is another kind of foot-pound from which the idea of motion is excluded, namely, the foot-pound of statical moment, or weight and leverage combined. Taking an ordinary scale-beam, with equal arms, the condition of balance or equilibrium is that the weights should be equal. But if the arms be unequal, then the product of the length of the arm and of the weight hanging to it must be the same for both weights, in order that they may balance one another. This product is called the *moment* of the weight about the point of suspension of the scale beam, and it is measured in foot-pounds, foot-tons, inch-pounds, and so forth. These foot-pounds must be carefully distinguished from foot-pounds of mechanical work.

The questions which arise on this subject are of the following character:

The scale-pan of a steelyard hangs 3 inches from the bearings, and the sliding weight weighs 1½ lbs., the bar just balancing the scale-pan. Where must the weight be put to balance 10 lbs. in the pan—that is, to weigh 10 lbs. of meat?

The moment of the meat will be 30 inch-pounds. The leverage of the weight to balance it will therefore be—

$$\frac{30 \text{ inch-pounds}}{1\frac{1}{2} \text{ lbs}} = 20 \text{ inches.}$$

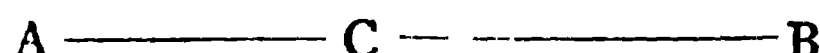
To weigh a heavy weight with the same steelyard, a fixed weight of 5 lbs. is slung 18 inches from the bearing link, on the side away from the scale-pan. Where must the sliding weight be put to balance 48 lbs. of beef?

The moment of the beef is 144 inch-pounds, and this is to be balanced by the moment of the fixed weight, $5 \times 18 = 90$ inch-pounds,

which leaves 54 inch-pounds unbalanced except by the moving weight. The length of lever for that will therefore be—

$$\frac{54 \text{ inch-pounds}}{1\frac{1}{2} \text{ lbs.}} = 36 \text{ inches.}$$

The foot-pound of leverage is connected with the foot-pound of work in this way. Let A and B be two weights fixed to a lever swinging about C,



and let us make the lever turn round C with any angular velocity whatever. Then the travel of A and B will be in the ratio of the lengths C A : C B, and the work done upon each to get the velocity will be in the ratio of A × C A : B × C B. If A × C A = B × C B, the work done upon each will be equal, and they will also balance statically. The conditions of statical equilibrium and of equality of mechanical work are thus necessarily expressed in the same denomination, namely, foot-pounds, or some other unit of length × weight. But there is actual motion in the one case, while in the other there is no actual motion, but only the condition that *if* a small motion be introduced, there shall be equality of work.

Examples.

1. A man weighing 160 lbs. carries a ton of coal up stairs to a height of 100 feet, by a hundredweight at a time. What is the amount of mechanical work done?

2. A full well, 350 feet deep and of 18 square feet sectional area, has to be pumped out. What is the work to be done?

3. A steam-engine has two cylinders, each with a piston area of 63·6 square feet, and 10 feet stroke, doing 25 strokes a minute, with 42 lbs. pressure per square inch. What is the I. H. P.?

4. Supposing it to pump 20,000 gallons of water 11 feet

high per second, what will be the efficiency of the pump and steam-engine together?

5. Four screw-jacks lift a 30-ton engine 6 inches in five seconds. What is the work done by each, and what power is applied to each, supposing ·4 of the work to be lost.

6. A 5 ton hammer falls through 5 feet and delivers 25 blows a second. What is the H. P.?

7. What is the mechanical work done in exhausting a receiver containing 10 cubic feet, the atmospheric pressure being 15 lbs. per inch?

8. A 600-pounder shot strikes an armour-plate with a velocity of 1,200 feet per second. Supposing the work to be equally divided between the shot and the plate, how much is expended in the heating and destruction of each?

9. To what H. P., acting for one minute, will this be equivalent?

10. A rifle shot weighing 530 grains is fired right upwards, and reaches a mile high. Neglecting the resistance of the air, what is the muzzle velocity? And what work has been done by the powder, supposing ·7 of it to have been utilised?

11. What work is required to bring to a standstill a railway train weighing 170 tons from a speed of 42 miles per hour?

12. The absolute weight of H.M.S. Agincourt, when laden, was 10,000 tons. Supposing her to have been going 5 statute miles an hour when she struck on the Pearl Rock, what work was expended in the blow?

13. What is the H. P. in a river a kilomètre wide, with a mean depth of 10 mètres, and a mean velocity of 5 kilomètres per hour?

14. A mill wheel with 40 feet fall is fed by a race of 10 feet sectional area, with a mean velocity of 6 feet per second. The mean velocity in the tail race is 10 feet per second, and the efficiency ·83. What is the effective H. P.?

15. A river at a certain point has the same mean speed as 15 miles below, namely 10 miles per hour. The difference

of level is 180 feet. What is the lost work per square foot of sectional area ?

16. A long pipe delivers with a velocity of 50 miles an hour with 120 feet head. What is the loss of head due to the resistance of the pipe ?

17. A steam fire-engine working at 165 effective H. P. delivers 1,815 gallons of water per minute through a given length of hose, with a certain nozzle, on the level. How many gallons will it throw if the branch is held at a height of 120 feet, and what is the effective area of the nozzle ?

18. A solitary wave enters the mouth of a harbour a mile wide. The sectional area of the wave (on a vertical section in the direction of its motion) is 5000 square feet, and the mean velocity of the particles 10 feet per second. What work must be done to bring it to rest ?

19. Supposing similar waves to follow one another at the rate of 10 per minute, what will be their H. P ?

20. A boy whirls a stone round his finger steadily in a vertical circle by a string 3 feet long. What is the difference of velocity at the highest and lowest points ?

21. A pound and a quarter hangs on one side of a pulley block and a pound on the other. Neglecting the weight of the string, friction, and so forth, what velocity will be acquired in 3 seconds from the time they are let go ? And what space will the weights have traversed ?

22. A rope is wound on a 16-inch capstan barrel which is driven by 12 handspikes with a leverage of 8 feet each. What pressure must be applied to each to weigh a 1-ton anchor ?

23. What will be the work done on each handspike to raise the anchor from 90 fathoms.

24. A steam crane is exactly balanced without its load or its engine, at a point 5 feet behind the pivot. The jib overhangs the pivot 10 feet. Supposing the crane to weigh 5 tons, and the engine and boiler 5 tons, where must the centre of the engine and boiler be placed to balance a load of six tons ?

25. It is estimated that a certain proposed shot, if fired vertically without resistance from the air, would reach a height of 22,000 feet. What is the muzzle velocity?

Answer. 1,190 feet per second.

26. Of two adjacent reservoirs, of 12 acres each, one is empty, and the other contains salt water 30 feet deep. How many foot tons of work are lost by allowing one to flow into the other? *Answer.* 3,360,343 foot-tons.

27. A pipe of $4\frac{1}{2}$ in. sectional area delivers water vertically downwards with a velocity of 48 feet per second from a tank fed by a stream. It works a turbine 100 feet below. The efficiency of the whole machine is ·55. What is the effective H. P.? *Answer.* 12·73.

28. An elbow-pipe is dipped into a stream running six miles an hour, the opening facing up-stream. How high will the water rise in the pipe?

Answer. 1·2 feet, very nearly.

29. The reciprocating part of an engine weighs 12 cwt., and makes 19 double strokes a minute, its highest velocity being 500 feet a minute. What is the whole H. P. which it is possible to lose by the reciprocation?

Answer. 3·3 H. P.

30. A water-plug bursts under a total pressure of 70 lbs. the square inch. With what initial velocity would it spirt against atmospheric pressure?

Answer. 90·4 feet per second.

31. At what velocity will a bridge weighing 1,400 tons be raised, supposing 100 effective H. P. can be applied to lifting it? *Answer.* 1·05 feet per minute.

32. A 30-ton hammer strikes 10 blows a minute with a velocity of 50 feet per second. Assuming three-fourths of the mechanical work to be utilised, or expended in friction, &c., how much H. P. is spent in shaking the foundation?

Answer. 198 H. P.

MENSURATION.

CHAPTER XVII.

SIMILAR FIGURES.

IN its most extended sense the word MENSURATION may be said to embrace the greater part of Mathematics. A complete treatise of it is therefore out of the question, especially in a work on Arithmetic; but it is thought desirable to give so much of it as can be used and understood with the assistance of a moderate knowledge of Geometry. This is the more necessary, because the best books on Elementary Geometry do not profess even to prepare the way for the application of Arithmetic. The principal applications of Mensuration are the following: 1. The measurement of lines, straight or curved, including the estimation of distances. 2. The measurement of flat surfaces or plane areas as they are otherwise called. 3. The measurement of the volume or space enclosed by curved surfaces, including incidentally the estimation of their weight from their shape and size. 4. The measurement of curved surfaces.

All mensuration rests on the fundamental proposition, that figures of the same shape have all their parts proportional. The sense in which this is to be understood may be very well exemplified by taking two cubes (which by virtue of being cubes are necessarily of the same shape), but of different sizes, and let us suppose that an edge AB of one of them is 1 inch, and that an edge of the other is 2 inches long. I then infer from Geometry the following propositions: 1st. That the length

of every line which can be drawn in one of them is exactly double the length of a corresponding line drawn in the other. For instance, the line AC drawn from corner to corner of one of the faces (called the diagonal of that face) of the smaller cube will have a length of 1·4142 in. The line drawn from one corner of the cube to its opposite corner A*a* (the *through diagonal*) will be 1·732 in. nearly. Therefore, as a consequence of the similar shape, the corresponding lines in the larger cube will each be *twice* this number of inches.

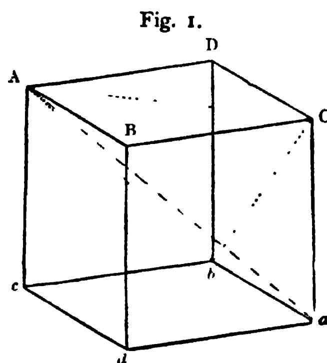


Fig. 1.

2nd. That any corresponding pieces of plane surface, bounded by corresponding lines in each of the two cubes, will be in the ratio of 2 × 2 : 1 or 4 : 1. For instance, any one of the plane faces in the larger cube will be four times the size of a plane face of the smaller cube. The same is true of any figure whatever drawn in the same way in each of the two cubes. Suppose, for instance, we cut each cube into two pieces through three corners, A, C, *b*; then the exposed surface of the triangle AC*b* will be four times as great in one cube as in the other.

3rd. The spaces contained within the surface of the two cubes will be in the proportion of 2 × 2 × 2 : 1 or 8 : 1, and this will be true, not only of the whole volumes of the cubes, but also of any similar portions of them. For instance, the pyramid ACD*b*, cut off by the plane section AC*b*, will be eight times as big in one cube as in the other; and if both cubes are cut out of one uniform material, such as soap or chalk, the corresponding piece from one cube will be eight times as heavy as the corresponding piece of the other.

We have taken 2 to 1 as the simplest illustration. But the proportion might have been any other. For instance,

if the edge of one cube be to the edge of the other in the ratio of $m : n$, then—

1. All corresponding lines are in the ratio of $m : n$.

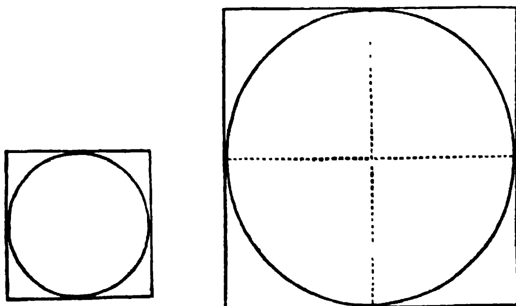
2. All corresponding portions of surface are in the ratio $m \times m : n \times n$, or $m^2 : n^2$, as it is more conveniently written.

3. All corresponding solid portions are in the ratio $m \times m \times m : n \times n \times n$, or $m^3 : n^3$, as it is written.

The reason of this is evident. If we imagine that one cube has expanded into the other by some process of stretching, the lines will only be altered in one dimension, viz. length. Surfaces will be altered in two dimensions, viz. length and breadth, and if each of these be doubled, the area will be increased 2 × 2 or 4 times. The solid has three dimensions, viz. length, breadth, and thickness (or depth or height), and if we increase the thickness also, it is clear that we must double again for the solid.

I will give another illustration of this principle. Suppose we have two circles, one struck with double the radius of the other, I say that the circumference of one is double that of

Fig. 2.



the other, and that the area of one is four times that of the other. The circle is an inconvenient figure to cut up, and I cannot show it by that process. I have therefore recourse to the following artifice. To each circle I circumscribe a

square ; that is, I draw a square so that the circle touches the four sides, as in the figure. Now without considering the actual lengths of the circle or their actual areas, it will probably be admitted (as easily seen) that the proportion between the areas of the circle and the square in which it lies is the same in both figures, and that in both figures the circumference of the circle bears the same proportion to the measurement round the four sides of the square. Now let us cut up the larger square by means of the dotted lines; we shall see that we have divided it into 4 pieces, each of which is equal to the small square. The areas of the squares are therefore in the proportion of 4 : 1 ; the areas of the circles being in the same proportion are also in the proportion of 4 : 1. On the other hand, the measurements round the two squares being each 4 times the side, are in the ratio of the sides, and the circumferences of the circles are also in the ratio of the sides. But the side of the circumscribed square is equal to the diameter of the circle, therefore the circumferences of two circles are in the same ratio as their diameters.[1]

The importance of the principle just stated and illustrated consists in this, that it enables us to separate that part of the measurement which depends upon shape from that part which depends upon size, and that when we have ascertained the former, a variation in absolute size is dealt with as a mere question of proportion, depending upon the accurate measurement of some corresponding line in each. Thus, if we find the number of cubic feet (say 60,000) contained in the hold of a vessel 200 feet long, and if we build another vessel 300 feet long, from the same drawings exactly, we know that since the lengths are in the proportion of 3 : 2, the hold of the larger vessel will contain ($\frac{27}{8}$ =) nearly $3\frac{1}{2}$ times as much as the other, or, more exactly, 202,500 cubic feet.

[1] **The above is intended as illustration, not as proof. A proof of a geometrical proposition would be out of place in a treatise on Mensuration.**

If, instead of giving the linear ratio between similar figures we give the ratio of any surfaces or areas corresponding in each, the linear ratio will be the square root of the surface ratio. If we want to obtain the volume ratio we must cube this. Hence the volume ratio is the cube of the square root of the surface ratio, or, what is the same thing, the square root of its cube. Thus, if of two exactly similar buildings, one covers 5 times as much ground as the other, their lengths will be in the proportion of $\sqrt{5} : 1$, and their solid contents or capacity in the ratio of $\sqrt{125} : 1$.

If the capacity or volume ratio be given, the lineal ratio will be as the cube root of that, and the surface ratio will be as the square of the cube root, or, what is the same thing, as the cube root of the square. For instance, a quart contains two pints. Hence, if the measures be of the same shape, their diameters will be in the proportion of the cube root of 2 to 1, that is, as 1·26 to 1 nearly, while the areas or surfaces will be in the proportion of the cube root of 4 to 1, or roughly of 8 : 5. But some care must be taken to note that similarity of shapes means that the heights are to be in the same proportion as the breadths. For instance, a pint pot and a quart pot, both of them round, and of the same height, are not of the same shape. The diameters in this last case would evidently be in the ratio of $\sqrt{2} : 1$, or roughly of 7 : 5, the bottom of one measuring in square inches double that of the other. In this case two dimensions only have been altered, the height remaining the same.

Examples.

1. The head of a 9-gallon cask is 11½ inches across, and its length is 21 inches, outside measure. What will be the corresponding measures in a 36-gallon cask?

2. What will be the contents of a cask of which the head diameter is 17¼ inches, and the length 31½ inches, and of proportionate thickness?

3. What do I mean by proportionate thickness? Suppose the 9-gallon cask to be an inch thick (average), what should be the thickness in question?

4. A certain ship, 200 feet long, is found to have 1800 tons displacement (or actual weight). What will be the displacement of a similar vessel 300 feet long.

5. Supposing that the smaller vessel takes 14 cwt. of paint to give it one coat, how much will the larger vessel take?

6. Two copper saucepans are of the same shape and weight, but one is half the thickness of the other. Supposing the smaller one to contain a gallon, how much will the other hold?

CHAPTER XVIII.

MEASUREMENT OF LINES.

1. *Straight Lines.*

A STRAIGHT line or combination of straight lines accessible throughout their whole lengths is measured by the direct application of a standard measure of length suitably divided, such as a two-foot rule or surveyor's chain. It often happens, however, in practice, that the lines are not accessible through part of their length, or else that they have no real existence, but simply indicate the shortest path between two points, between which there is no actual direct connection. The line which we may have to measure may be either a wire strained across a pond or the mere distance between the points of two neighbouring steeples. In these cases the artifice which we resort to practically amounts to this. We connect the line, which we have to measure, but cannot reach, with some other line which we can reach, by means of a system of lines so chosen that we can either by geometrical or by arithmetical methods find the ratio which the

length of the line which we wish to measure bears to the line which we *can* measure. It is then a mere question of proportion to determine the required line. An example is to be found in the problem of measuring the height of a tall flagstaff by means of its shadow on level ground. We first measure the shadow with a measuring tape. We then set a walking-stick upright, and measure both its length and that of its shadow. Then we know that the length of the flagstaff bears the same proportion to the length of its shadow as in the case of the walking-stick. For instance, if my walking-stick is 35 inches, and throws a shadow of 25 inches, and if we have found the shadow of the flagstaff to be 50 feet long, we know that the height of the flagstaff will be 70 feet, because the triangles made by the stick, its shadow, and the ray of light which joins the top of the stick to the extremity of the shadow, are similar, both for the walking-stick and flagstaff. Hence we have simply the proportion, 25 in. : 35 in. :: 50 ft. : 70 ft. The principle here illustrated is that on which all trigonometrical measurement depends. We have taken the simplest case. Those who desire to study this kind of measurement should read the Treatise on Trigonometry. There are, however, some details about the foregoing process which require a little attention, especially with regard to the assumptions made in order to apply our arithmetic. We assume the flagstaff to be upright, and that we know how to put the stick upright too. It is not easy in practice to make quite sure of these things. We assume that the top of the flagstaff has a well-defined point, and that our measurement is taken from the middle line of the mast. We also assume that the tops of the flagstaff and of the walking-stick cast a well-defined shadow which we can measure accurately. Now this is never the case. For the lower part of the disc of the sun or moon throws its shadow farther back than the upper part, and this leaves a band of partial shadow gradually passing from full shade to full light. All

these difficulties must be got rid of in some way or other if accuracy is desired. My object, however, is not to show how this somewhat clumsy mode of measurement can be made accurate, but to call attention to the marked difference that there is between applying our arithmetic to *things* and to questions. In the latter case the figures are given us, and we take them to be accurate; in the former a number of little *reductions* (as astronomers call them) have to be made before we can even start our arithmetic. But these corrections, which are different for nearly every different business to which measurement can be applied, will not come within the object of a general treatise on Mensuration.

These remarks apply even to such a thing as the mere measuring of an iron tank. It is very seldom that the corners are as truly angular, as they are in a slate cistern, and the sides are not always true. In fact, there are very few things so accurately shaped, and sharply defined, as to be capable of being exactly measured by the mere application of a carpenter's rule. Even where a scale can be applied fairly, the measurement, for minute philosophical purposes, is still far from accurate. Further explanations as to the indirect measurement of lines will be found in the Treatises on Practical Geometry and Trigonometry.

The relation between the sides of a right-angled triangle considered without reference to either of the acute angles falls within the rules of common arithmetic, seeing that the square on the hypothenuse is equal to the sum of the squares on the other two sides. If, therefore, the other two sides be given in linear units, we have simply to square them, add them, and take the square root. If, on the other hand, the hypothenuse and one other side be given, we have to subtract the squares and take the square root, or we may use the following equivalent rule:

Either of the sides adjacent to the right angle is a mean proportional between the sum and difference of the other two sides.

2. *Circles.*

It has been already stated that the circumferences of circles are in the same ratio as their diameters. It therefore becomes of importance to know what is the constant ratio of the circumference of a circle to its diameter. It is in fact an interminable decimal, and it has been shown that it cannot be exactly expressed by the ratio to one another of any two finite numbers, however great. Archimedes showed that it lay somewhere between $\frac{22}{7}$ and $\frac{31}{10}$. Adrian Metius gave as a much nearer approximation, $\frac{355}{113}$, and a geometrical approximation frequently used amounts to the arithmetical expression $3 + \frac{1}{10}\sqrt{2}$. None of these, however, are accurate. It has been calculated as far as 441 decimal places by two independent calculators, and may be regarded as known with certainty to that extent of accuracy.[1] To save the trouble of repeating a long series of figures, it has been generally agreed among mathematicians to express this ratio by the small Greek letter π. We give its value to 20 decimal places, which is 16 more than are ever used in practical mensuration:

$$\pi = 3{\cdot}14159\ \ 26535\ \ 89793\ \ 23846$$

For most purposes it is near enough to take $\pi = 3{\cdot}1416$ and $\frac{1}{4}\pi = {\cdot}7854$. There is no other curved line of which the length is practically useful in common measurements.

The measurement of circular arcs depends upon the geometrical principles, that

(1) In equal circles the arcs are proportional to the angles which they subtend.

(2) In different circles the arcs which subtend the same angle are proportional to the radii from which they are struck.

It may be convenient also to call to mind that the angle subtended by an arc from any point of the circumference is

[1] See the 'English Cyclopædia,' Article Quadrature.

half the angle at the centre of a circle. Hence, if an angle is given in degrees, say x degrees, the arc subtending that angle, and struck with a radius r, is $\frac{r \pi x^\circ}{180}$. If the angle be given in minutes x', the arc is $\frac{r \pi x'}{10800}$, and if in seconds x'', the arc is $\frac{r \pi x''}{648000}$.

			Reciprocals
π	=	3·1415926536	·3183098862
$\frac{\pi}{180}$	=	·01745329252	57·29577952
$\frac{\pi}{10800}$	=	·0002908882c87	3437·746771
$\frac{\pi}{648000}$	=	·000004848136812	206264·80625

Thus, if the radius be 54 inches, the arc subtending an angle of 22° 30′ is

$$22{\cdot}5 \times 54 \times {\cdot}0174533 = 21{\cdot}20575$$
$$\text{or } 1350 \times 54 \times {\cdot}000290888 = 21{\cdot}20575$$

If the arc A be given in length, the angle in degrees is evidently $x^\circ = \frac{180}{r \pi}$ A and in minutes $\frac{10800}{r \pi}$ A, and so on. For example, the arc which measures 36 inches with a radius of 54 is—

$$\frac{180 \times 36}{54 \pi} = \frac{120}{\pi} = 38{\cdot}1972^\circ = 38^\circ\ 11'\ 50''.$$

Again, if the angle and length of arc which it subtends be given, we shall find the radius by means of the formula

$$r = \frac{180}{\pi} \frac{A}{x^\circ} = \frac{10800}{\pi} \frac{A}{x'}, \text{ \&c.}$$

For instance, if an arc of 6 feet subtends an angle of 10°, we shall have—

$$r = \frac{180}{\pi} \frac{6}{10} = {\cdot}6 \times 57{\cdot}29578 = 34{\cdot}3775 \text{ feet.}$$

Care must of course be taken not to confuse denominations,

as by taking the radius in feet and the arc in inches. If the question comes up in such a form, the proper reduction must be made.

3. *Other Curves.*

It is not possible to lay down any purely arithmetical rule for ascertaining the lengths of curves in general from a mere knowledge of their properties without actual measurement.

A curve drawn on a piece of paper can be measured with a great approach to accuracy by dividing it into short lengths and then setting off these lengths with the compasses on a straight line as if they were straight pieces. This of course fails where the curvature is very great, but a little geometrical dexterity soon disposes of such a case. The character of the approximation which we use consists in the substitution of the chord for the curve. Let us draw at A and B lines touching the curve and meeting, let us suppose, in C. Then it is evident that the length of the curve APC will be intermediate to the direct line AB and the broken line ACB. This affords us a means of determining the limit of error. Let fall the perpendicular CD on AB. To treat the question arithmetically, we will now suppose that AD is 6 inches, DB 4 inches, and CD 2 inches. Then, by the rule for right-angled triangles, the broken line ACB will be, in inches, $\sqrt{40} + \sqrt{20}$ or (nearly) $6{\cdot}325 + 4{\cdot}472 = 10{\cdot}797 = 10{\cdot}8$ (nearly). Hence, in this which must be regarded as an extreme case, the difference (·8) between the broken line ACB and the line AB is about $\frac{1}{12}$ of the latter, and the error in the length of the curve will be somewhat less. Next suppose that the curve is so much flatter that the line CD only measures 1 inch instead of 2. Then the broken line ACB will be $\sqrt{37} + \sqrt{17}$ or $6{\cdot}083 + 4{\cdot}123 = 10{\cdot}2$ nearly, and the error is now about $\frac{1}{50}$th. This shows how rapidly the error diminishes as the curve flattens, or, what comes to the

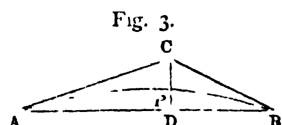


Fig. 3.

same thing, as we measure the curve in short stretches instead of long ones. Another practical way of measuring a curve upon paper is to take a straight-edged strip of paper about $\frac{1}{8}$th of an inch wide. This, laid flat upon the drawing, can easily be made to fit the curve, the extremities of which may be marked off upon it with a fine pencil; on straightening it the length may be measured off on a scale. This method is commonly used by draughtsmen, a flexible batten being sometimes used instead of the paper.

Examples.

1. Find the diagonals of all the faces, and also the *through diagonals* of a rectangular block, 12 × 4 × 3 feet.

2. The diagonal of a rectangle is 25 feet, and one side is 7 feet. What is the other side?

3. The diagonal of a square is 512 feet. What is its side?

4. The diagonal of a cube is 13·69 mètres. What is the volume?

5. Given the sum and difference of the sides of a right-angled triangle as 23 yards and 7 yards, find the hypothenuse.

6. The base of a right-angled triangle is 100 yards; and the difference between the other side and the hypothenuse is 40 yards. Find the lengths of the other side and of the hypothenuse.

7. What number of degrees, &c., is contained in an arc equal to once and a half the radius?

8. The diameter of a circle is one mètre. What is its circumference in inches?

9. What is the length of an arc of 54° in a circle whose radius is 27 inches?

10. An arc of 72° measures 30 inches. What is the radius of the circle?

11. What is a minute of longitude on the equator, supposing the diameter of the earth to be 41,847,662 feet?

12. What is the proportionate error of the following rough rule? To find the diameter of a shaft in inches, measure its circumference in centimètres, and divide by 8.

CHAPTER XIX.

MEASUREMENTS OF AREAS.

THE area most easy to be measured is what is called a rectangle, that is, a four-sided figure having its opposite sides parallel to one another and all its angles right angles. Its area in square units is the product of the linear units in its length and breadth severally. If the ratio between the length and breadth is a finite ratio, this can be seen by actually cutting up the rectangle into small squares and counting them. For instance, if one side be $\frac{3}{2}$ inches and the other $\frac{7}{2}$, we may take $\frac{1}{2}$ inch for the unit of length and cut up the rectangle accordingly. We shall evidently get 3×7 squares, each $\frac{1}{2}$ in. $\times$ $\frac{1}{2}$ in. Four such go to the inch. Hence the area is $\frac{3 \times 7}{2 \times 2}$ square inches.

When the sides are incommensurate, that is, when their ratio cannot be expressed by any finite fraction, this demonstration does not apply. For instance, suppose the breadth is 1 in. and the length in inches $\sqrt{2} = 1\cdot4142\ldots$ which is an interminable decimal. We can find no unit, however small, to which the foregoing construction would apply. But if we stop at any given decimal, say the second, we see that the value of the fraction lies between $\frac{141}{100}$ and $\frac{142}{100}$. Now if we take $\frac{1}{100}$ in. as the unit we can apply the proof to both these fractions, and the rectangles which these make only differ by a strip 1 in. broad, and $\frac{1}{100}$th of an inch in width. Next, let us go a step further, and remarking that the $\sqrt{2}$ lies between $\frac{1414}{1000}$ and $\frac{1415}{1000}$, we see (1st) that the proof may be applied to rectangles of this length by taking $\frac{1}{1000}$ in. as our unit, and (2nd) that these rectangles, between which our actual rectangle lies in respect of size, only differ

by a strip 1 in. long and $\frac{1}{1000}$ in. wide; and we have thus only to take more decimals to make this difference as small as we please. We are consequently in a position to prove the statement generally, apart from a small error which we can make as small as we please, by simply taking decimals enough, and, inasmuch as the truth of the proposition does not depend upon the number of decimals which we have either the time or the industry to take, we infer it to be actually true for incommensurate numbers as well as for finite ratios.[1] We may therefore take it universally that the area in square units is equal to the product of the length in linear units and the breadth in linear units. It is on this proposition that the measurements of our every-day life depend; everything sold by the yard, by the foot, or by the inch, whether the length alone varies, or whether the breadth varies too, really depends upon this principle.

The area of a parallelogram which is not right-angled is easily obtained from this consideration: That we can convert it into a rectangle by cutting off a triangle at one end and putting it on at the other. Its area is thus seen to be the length multiplied into the breadth measured perpendicularly, or, as it is more commonly stated,

Fig. 4.

C D

A a B b

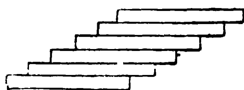
$$\text{Area} = \text{base} \times \text{altitude}.$$

There is another way of viewing this problem which is very important to be understood, because it is the only principle which we are able usefully to apply to the measurement of certain solids, and as plane geometry is more easy to comprehend than solid, it is best illustrated by a question of areas.

[1] The reasoning here is worth the attention necessary to understand it. It is in point of fact only by reasoning which is equivalent to this that we can justify the use of such quantities as $\sqrt{2}$, π, and so forth, according to the ordinary rules of arithmetic.

Let us imagine that a rectangle is cut up into slices by a series of equidistant lines parallel to one pair of sides, and then that the slices are made to slide one upon the other so as to form a regular flight of steps. Then if we are at liberty to neglect the roughness of the edges, we have converted the rectangle into a parallelogram of the same area, and this area is measured by the product

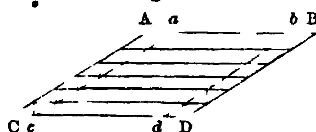
Fig. 5.



base × altitude.

We have still to show that we are at liberty to neglect the roughness of the edges. For this purpose draw an oblique line on each side through the inner edges of the steps and also through their outer edges. We shall thus get an outer parallelogram, ABCD, and an inner parallelogram *abcd*, one of which will be greater and the other less than the original rectangle: as we increase the number of steps by making the slices thinner and more numerous, it is evident that the parallelograms, ABCD and *abcd*, tend to coincidence with one another and therefore to equality with the original rectangle. If therefore we imagine the subdivision to be indefinitely increased, so that the edge of the steps practically merges into the straight line AC or *ac*, the error, if any, disappears, and we get another proof of the proposition that the area of a parallelogram may be measured by the product—base × altitude. Both in plane and in solid it is easy to see that this affords proof that the irregularity of the edges may be neglected without risk of error, if we simply take the slices sufficiently thin, and, as the division is merely imaginative, we may do so without any limit. We have stated this process with some care on a case too simple to need its application, because we shall

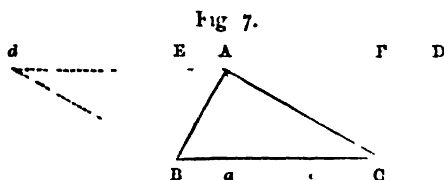
Fig. 6.



want hereafter to apply it to cases in which no other method applies.

Triangles.—A triangle may be considered as half a parallelogram. For instance, the triangle ABC may be considered as half of either of the parallelograms, ABCD, ACB*d*, or EBCF

Fig 7.



The area of the triangle is thus seen to be half the product of the base and altitude.

Fig 8

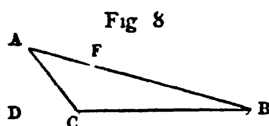
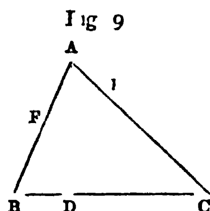


Fig 9



Either side may be taken for the base

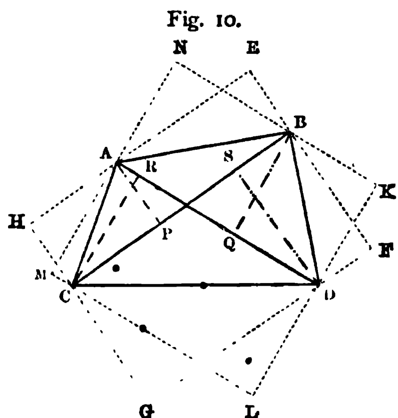
Thus the area of the triangle ABC may be expressed indifferently as $\frac{1}{2}$AD . CB, $\frac{1}{2}$BE . CA, or $\frac{1}{2}$CF . BA.

In a right-angled triangle the perpendiculars from two of the angles are already drawn, and the area is simply half the rectangle between the two sides which contain the rectangle, or, speaking arithmetically, half the product of their lengths.

Any quadrilateral or trapezium may be measured by drawing a diagonal and letting fall perpendiculars upon it from the other two angles. This is simply equivalent to considering the quadrilateral ABCD as equal to one-half of either of the rectangles EFGH or KLMN. By this construction it

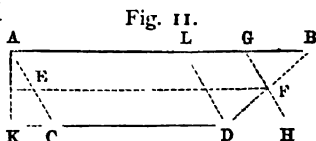
is evident that the area of ABCD may be written indifferently as ½AD (BQ + CR), or ½BC (AP + DS).

Fig. 10.



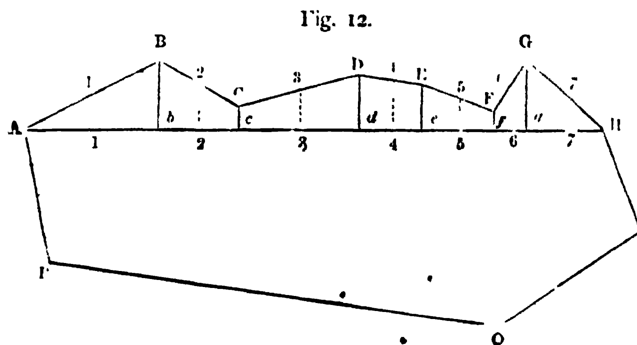
Trapezoid.—There is a simplification when two sides of the quadrilateral happen to be parallel, and as the case very frequently occurs in practice, it is necessary to explain it fully. Let ABCD be a trapezoid of which the sides AB and CD are parallel. Let E and F be the middle points of the other two sides AC, BD. Also let AK be the breadth measured perpendicularly to the parallel sides ABCD. Then I say that the area ABCD is equal to AK . EF or ½AK (AB + CD). For through F and D draw HG and DL parallel to CA and produce CD to H as in the figure. We have now converted ABCD into the parallelogram AGHC by cutting off the triangle BFG and adding on the equivalent triangle DFH, and since the area of a parallelogram = base × altitude, the area is EF × AK. Further, since AL = CD and LG = GB, it is easily seen that AG (which equals EF) is a mean between AB and CD. This shows that the area may be written as ½AK . (AB + CD).

Fig. 11.



It is on this proposition that measurement by offsets depends. Suppose we have to measure a field of which (to

Fig. 12.



take a simple case) four sides are straight lines but the fifth side is replaced by the broken line ABCDEFGH. We first run a convenient *base line* AH, so as to cut off the irregular part from the rest; the regular part being measured by some of the methods already explained, we *set off*, as it is called, all the points ABCDEFGH by lines drawn through them perpendicularly to the base line. The lines B*b*, C*c*, D*d*, &c., are called *offsets*. We have now cut up the irregular part into trapezoids, of which the offsets are the parallel sides and the distances *ab*, *bc*, &c., the perpendicular breadths. We may, it is true, have a triangle or two, but these may be considered as trapezoids, where one of the offsets = 0, and so their occurrence does not confuse the rule.

We have now to calculate separately the areas of the trapezoids (or triangles, if any) AB*b*, *b*BC*c*, and so forth, by multiplying the base of each into the mean of the offsets by which it is terminated. Thus, area $b\text{BC}c = bc \times \frac{1}{2}(\text{B}b + \text{C}c)$. We then add them all together to get the total area above the base line AH.

Thus, suppose the distances between stations and offsets at each to be as follows :

	A	*ab*	*bc*	*cd*	*de*	*ef*	*fg*	*gh*
Distances between stations in feet,	0	17	10	15	8	9	14	10
Offsets,	0	8	3	6	5	2	9	0
	A	*b*B	*c*C	*d*D	*e*E	*f*F	*g*G	*h*H

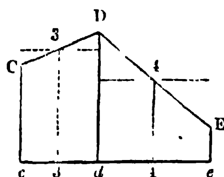
we get the following work:

$$\tfrac{1}{2}(0 + 8) \times 17 = 68$$
$$\tfrac{1}{2}(8 + 3) \times 10 = 55$$
$$\tfrac{1}{2}(3 + 6) \times 15 = 67{\cdot}5$$
$$\tfrac{1}{2}(6 + 5) \times 8 = 44$$
$$\tfrac{1}{2}(5 + 2) \times 9 = 31{\cdot}5$$
$$\tfrac{1}{2}(2 + 9) \times 14 = 77$$
$$\tfrac{1}{2}(9 + 0) \times 10 = 45$$

Total area above A H 388 square feet.

The arithmetical work might be shortened if instead of taking the offsets at the ends of the straight lines ABCD, we had taken them at the middle points of these. If the lines AB, &c., are straight, this merely amounts to getting the means of the previous offsets by a geometrical instead of an arithmetical process. In fact it is evident that the line 22 in the drawing is $\frac{1}{2}$ (B*b* + C*c*).

Fig. 13.



The general principle may perhaps be more clearly understood by considering the following figure, which may be regarded as representing a small part of the previous figure. Evidently the area *c*CDE*e* may be represented indifferently as the sum of

$$cd \times \tfrac{1}{2}(\mathrm{C}c + \mathrm{D}d) + de \times \tfrac{1}{2}(\mathrm{D}d \times \mathrm{E}e)$$

or, as $cd \times \text{line } 33 + de \times \text{line } 44.$

There is a marked advantage in having the stations *a*, *b*, *c*, &c., equidistant from one another, because then we may perform a single multiplication after the addition, instead of using separate multipliers before the addition. What we then do is in effect to find the length of a ribbon

whose area is that of the irregular space to be measured, and whose breadth is the uniform station distance, *ab*, or *bc*, or *cd*, &c. This advantage becomes still more marked when the lines AB, BC, &c., are not straight but nearly so, and when thus the process is approximate instead of exact.

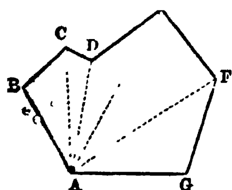
In actual surveying, it is usual to mark down in a field-book the distances of all the stations at which offsets are taken from a starting point. These are arranged in a column in the middle of the page, beginning from the bottom, and the offsets are entered right and left as the case may be. The field-book for the last example would be as follows:

	B	
	73	
	63	9
	59	2
	50	5
	42	6
	27	3
	17	8
From	A	

This supposes that the offsets are all taken to the right of the base line. The work of finding the area is the same as before. That of course would not be affected by their being all to the right instead of to the left.

A figure made up of straight lines may also be measured by cutting it up into triangles and measuring each triangle separately. This is best done in general by drawing lines from one corner of the figure to every other corner of the figure. This saves two triangles. It may happen that one or more of the triangles may have to be subtracted instead of added, as the triangle ACD in Figure 15. It is also possible and sometimes convenient to obtain the area of a polygon by means of offsets.

Fig. 14.



There is no difficulty in doing this if we take notice of change of direction. Thus the polygon ABCDEFGH may be measured by adding and subtracting properly the trapezoids AB*ba*, BC*cb*, &c., to find which are to be subtracted, it is only necessary to go round the polygon in one specific order and mark as subtractive all those for which we have to 'go backwards.' Thus, if we take the order ABCDEFGHA, and call *ab* 'going forward,' *cd*, *de*, and *fg*, also 'go forward.' The trapezoids standing on these

Fig. 15.

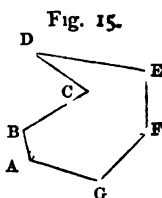
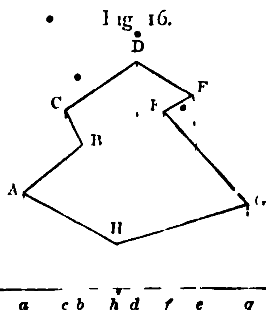
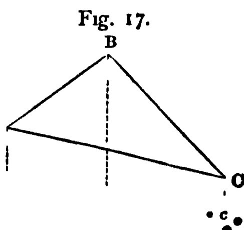


Fig. 16.



bases will therefore be additive, but since *bc*, *ef*, *gh*, and *ha* go *backwards*, the trapezoids standing on these must be subtracted. Now, measuring these trapezoids by the rules already given, and performing the additions and subtractions just indicated, we shall get the area of the polygon. Perhaps this will be better seen by applying it to a simpler case. The triangle ABC will clearly be the difference between the areas of the polygons *a*ABC, and *a*AC*c*. The first polygon splits up into the two trapezoids AB*ba* and BC*cb*. Hence finally, the area of the triangle is

Fig. 17.



AB*ba* + BC*cb* − CA*ac*.

Regular Polygons.—These have all their sides of equal length and all their angles equal. It is always possible to find a circle which shall pass through all their corners or angles, called the circumscribed circle, and another the inscribed circle, which touches all their sides at their middle points. The centres of both these circles are in the same point. If we join this centre to every angle of the polygon, we shall cut it up into a lot of isosceles triangles, all equal and similar. Hence we can find the area of the regular polygon by simply finding the area of any one of these triangles and multiplying it by the number of sides. The area of the triangle is best found by letting fall a perpendicular from the centre on the side. This perpendicular is simply the radius of the inscribed circle, and the area of the polygon is thus found to be the product of this radius by half the sum of the sides.

This last property is not peculiar to regular polygons, but is true of any polygon in which a circle can be inscribed so as to touch all its sides.

The measurement of regular polygons, however, belongs to Trigonometry.

I take the following table from Dodson's 'Calculator':

Regular Polygons.

Number of sides	One side in terms of R. of circumscribed circle	One side in terms of R. of inscribed circle	Area in terms of square on side
3	1·7320508	3·4641016	·4330127
4	1·4142136	2	1
5	1·1755705	1·4530851	1·7204774
6	1	1·1547005	2·5980762
7	·8677674	·9631491	3·6339124
8	·7653668	·8284271	4·8284271
9	·6840403	·7279405	6·1818242
10	·6180340	·6498394	7·6942088
11	·5634651	·5872521	9·3656404
12	·5176381	·5358984	11·1961524

Examples.

1. (1) The base of a triangle is 300 yards and its altitude 150. Find the area.

(2) The two diagonals of a quadrilateral field are at right angles to one another, and measure 350 and 200 yards respectively. What is the area?

(3) The diagonal of a quadrilateral field is 550 feet, and the offsets from it to the other two corners are 150 feet and 120 feet. Find the area.

(4) Two opposite sides of a field are parallel, 140 yards and 170 yards long. The measure across the field is 90 yards. What is the area?

(5) A polygonal field is measured by running a base line from one corner A to another B, and taking offsets right and left, as in the following 'field-book':

	Yards	
	250	B
40	220	
	180	60
90	160	
	150	70
	90	20
30	70	
	50	15
From	A	go N.W.

Find the area.

(6) See question (3). Why are the offsets only and not the stations given in this case? Make out the field-book on the supposition that the stations are at 200 and 350 feet from one end.

2. (1) A rectangular field measures 172 yards by 93. What is the length of a diagonal, and what is the area?

(2) The diagonal of a rectangular field is 440 yards, and one side measures 264 yards. What is the area?

(3) A ladder 26 feet long is set against a wall with its foot 10 feet out. How high will it reach?

(4) A rectangular tank measures 3 × 4 × 12 feet. What is the distance from any corner to the opposite corner?

(5) A slate ridge-roof with eaves is 50 feet long, 30 feet wide, and 10 feet pitch. What is the quantity of slating in square yards?

(6) A ridge-roof with sloping ends is 50 feet long, 20 feet wide, and 10 feet pitch, and the ridge is 30 feet long. What is the superficial measurement?

(7) The two sides of a right-angled triangle about the right angle are 15 yards and 8 yards. Find the perpendicular on the hypothenuse.

(8) OP and OQ are two lines meeting in O. At a point A on OP, 3 inches from O, AB is drawn, perpendicular to OP, meets OQ in B, and is found to measure 4 inches. From B draw BC perpendicular to OQ meeting OP in C. From C draw CD perpendicular to OP, meeting OQ in D. What is the distance OD?

3. (1) ABCDEFG is a polygon.

AB = 5 in.				
AC = 10 in.	Perp. from	B on AC	=	3 in.
AD = 12 in.	,,	C on AD	=	4 in.
AE = 14 in.	,,	D on AE	=	2 in.
AF = 8 in.	,,	E on AF	=	2 in.
AG = 4 in.	,,	F on AG	=	4 in.

All the perpendiculars are on the same sides of the lines to which they are drawn at right angles. Find the area and the lengths of the other sides.

(2) ABCDEFG is a polygon circumscribed to a circle, 1 foot in diameter, and its perimeter is 4·5 feet. What is its area?

(3) Two sides of a triangle circumscribed to a circle 8 inches in diameter measure 20 and 12 inches respectively.

The area of the triangle is 96 square inches. What is the length of the third side?

(4) O is a point inside a convex polygon, ABCDEFGH.

AB = 10 in.	Perp. from O on AB =	5 in.
BC = 12 in.	" " BC =	14 in.
CD = 8 in.	" " CD =	26 in.
DE = 10 in.	" " DE =	34 in.
EF = 12 in.	" " EF =	22 in.
FG = 11 in.	" " FG =	16 in.
GH = 9 in.	" " GH =	9 in.
HA = 7 in.	" " HA =	3 in.

Find the area.

(5) The following is a field-book of the polygonal figure ABCDEFGHKLMNOPA. Find the area.

To L 20	135	
	122	8 to K
	115	10 to H
To M 50	110	
	100	70 to G
To N 25	75	
	70	15 to F
To O 40	60	
To P 50	53	
	45	50 to E
	30	38 to D
	22	25 to C
	10	40 to B
From	A	go north

(6) Find the area of a polygonal figure, ABCDEFGHK, by offsets taken from a base quite outside it, according to the following field-book:

	130	50 to F
	110	140 to C
	90	30 to G
	84	90 to E
	80	100 to D
	40	10 to H
	35	80 to K
	30	150 to B
om	a	100 yards to A

The student will do well to draw the figures in the last three examples to scale.

4. (1) Taking the circumference of a circle in terms of the diameter as 3·141592, what errors are made by taking it as equal to the perimeter of

(*a*) The inscribed hexagon i;
(*b*) The circumscribed hexagon c;
(*c*) Their mean $\frac{1}{2}i + \frac{1}{2}c$;
(*d*) $\frac{2}{3}i + \frac{1}{3}c$?

(2) Verify the value of the area of the octagon in terms of the side, by considering it as a square with the corners cut off.

(3) Taking the area of a circle in terms of the diameter as ·785398, what errors are made by taking it as equal to

(*a*) The inscribed hexagon I;
(*b*) The circumscribed hexagon H;
(*c*) The mean between them $\frac{1}{2}\text{I} + \frac{1}{2}\text{H}$;
(*d*) $\frac{1}{3}\text{I} + \frac{2}{3}\text{H}$?

(4) Find this for the dodecagon or regular figure of 12 sides.

(5) What is the area in acres of an octagon whose side is 100 yards?

(6) A regular polygon of 11 sides measures 500 acres. What is its perimeter?

5. Make out for yourself a table of the regular polygons from 3 to 12 sides with the following headings :

(1) Number of sides.
(2) Perimeter in terms of radius of circumscribed circle.
(3) Perimeter in terms of radius of inscribed circle.
(4) Area in terms of square on perimeter.
(5) Perimeter in terms of square root of area.
(6) Area in terms of square on radius of circumscribed circle.
(7) Area in terms of square on radius of inscribed circle.

Measurement of Curved Areas by means of Offsets.

Consider the area as having a flat base. Divide that base into an even number of equal parts and run offsets from the odd points (as numbered in the figure). Add all these offsets together, either by measuring them separately and adding them, or, what is simpler, doing the addition mechanically, by marking them off one after the other on a straight-edged slip of paper. The approximate area will be the sum (or total length on the slip) multiplied by the interval between the ordinates. Thus the area between the curve and the base line is $x(\text{A}a + \text{B}b + \text{C}c + \text{D}d)$. For suppose we draw all the offsets and through the points A B C and D draw lines parallel to the base meeting the even offsets (part of such a figure is here represented), it is evident that if we can consider the short bit of curve between the even ordinates as a straight line, this simply falls within the rule for measuring a trapezoid. But, inasmuch as the top of the trapezoid is not straight, it is approximate and not accurate. Nevertheless the accuracy is very considerable if

Fig. 18.

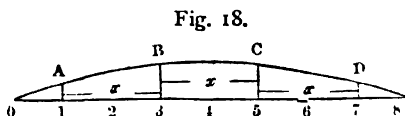
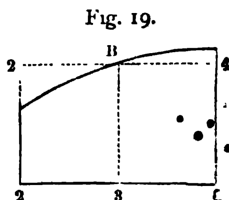


Fig. 19.



a sufficient number of ordinates be taken, and it can be proved that doubling the number of offsets quarters the error. The process fails to be accurate when the curve cuts the base at right angles or nearly so. If we were to apply it to a semicircle, for instance, we should not get a good result, even if we took a great number of offsets. These offsets are generally called ordinates by mathematicians, but surveyors and engineers call them offsets. If you wish to be very exact without measuring many ordinates you must proceed as follows: Divide the base as before into an even number of equal parts, and draw *all* the ordinates. Join the heads of the even ordinates as numbered in the figure by a straight line. Divide the piece of the odd ordinate intercepted between this straight line and the curve into three equal parts (by guess) and mark the division nearest the curve. Now, instead of measuring the odd ordinates from the base to the curve, measure them from the marked division. The sum of these corrected ordinates multiplied by the interval between them (or double the interval between the divisions of the base) will give the area. Thus, in Fig. 20, the area PRASBTQ is $x \times (a\alpha + b\beta)$. But I consider it preferable in practice to increase the number of divisions rather than to have recourse to this correction. If a closed curve is to be measured by this means, it must be cut up into convenient segments, and each part measured separately. A similar device must be had recourse to when the curve cuts the base nearly at right

Fig. 20.

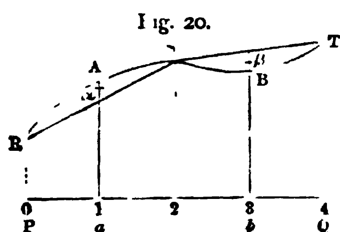
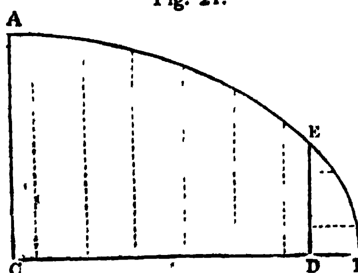


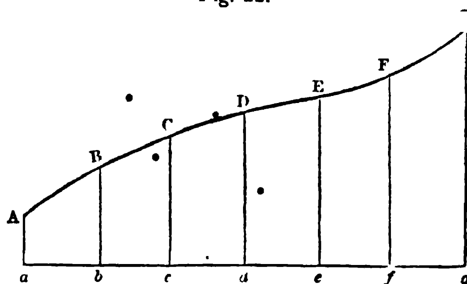
Fig. 21.



angles. For instance, the area ABC may be conveniently measured by cutting off the portion DBE, and then measuring separately the portion ACDE by offsets on CD, and the portion DBE by offsets on DE.

In many cases, however, we cannot take exactly what ordinates we please, and it is a more common thing to give the equidistant ordinates in offsets as follows:

Fig. 22.



In this case, if we neglect the curvature of the small arcs AB, BC, CD, &c., we have

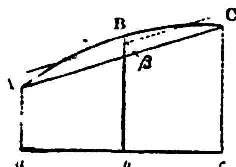
$$\begin{array}{lll}
\mathrm{A}b\mathrm{B} = \tfrac{1}{2}\mathrm{A}a + \tfrac{1}{2}\mathrm{B}b & & \times ab \\
\mathrm{B}bc\mathrm{C} = \quad \tfrac{1}{2}\mathrm{B}b + \tfrac{1}{2}\mathrm{C}c & & \times ab \\
\mathrm{C}cd\mathrm{D} = \quad\quad \tfrac{1}{2}\mathrm{C}c + \tfrac{1}{2}\mathrm{D}d & & \times ab \\
\quad \&c. \quad\quad\quad \&c. & & \\
\mathrm{E}ef\mathrm{F} = \quad\quad\quad \tfrac{1}{2}\mathrm{E}e + \tfrac{1}{2}\mathrm{F}f & & \times ab \\
\mathrm{F}fg\mathrm{G} = \quad\quad\quad\quad \tfrac{1}{2}\mathrm{F}f + \tfrac{1}{2}\mathrm{G}g & & \times ab \\
\hline
\text{Area} = (\tfrac{1}{2}\mathrm{A}a + \mathrm{B}b + \mathrm{C}c + \mathrm{D}d + \&c.\,.+ \mathrm{E}e + \mathrm{F}f + \tfrac{1}{2}\mathrm{G}g) & & \times ab
\end{array}$$

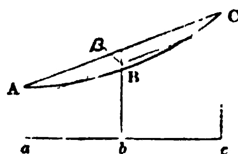
The rule is evident. Take half the first and last ordinates, and add their sum to the sum of all their intermediate ordinates taken singly. Multiply by the interval.

It is in this way that French engineers and shipbuilders always calculate curved areas. The English use another mode, by which, when the curvature is considerable, rather more correctness is obtained with the same number of ordinates. But the French consider that the best means of ob-

taining correctness is to increase the number of ordinates rather than to complicate the rule, and I agree with them. The English method depends upon a property of the parabola, which is as follows. Let ABC be an arc of a parabola

Fig. 23.





whose axis is parallel to the offsets A*a*, B*b*, and C*c*. Join AC by a straight line and produce *bc* to meet it, divide the portion of *b*B intercepted between AC and the curve into three equal parts and call the division nearest the curve β. Then, by a well-known property of the parabola, we may express the curved area *a*ABC*c*

Arithmetically as $\frac{1}{3}$ (A*a* + 4 B*b* + C*c*) × *a b*,
Geometrically as β *b* × *a c* = β *b* × 2 *a b*.

I leave it to the student as an easy exercise to show that these are equivalent, namely, that β*b* is necessarily $\frac{1}{6}$(A*a* + 4 B*b* + C*c*).

This settled, the English practice is to take always an odd number of offsets, or, what is the same thing, an even number of intervals. We then consider the curve which connects the heads of three offsets as being a parabola. We use the principle just stated to find the area of each portion including three offsets, and add them together. In practice, as the multiplication by the interval and the division by 3 run through the process, we perform the additions first. Thus, if the offsets be *a*, *b*, *c*, *d*, *e*, *f*, &c., we have for the

Areas from	
a to *c*	$a+4b+c$
c to *e*	$c+4d+e$
e to *g*	$e+4f+g$
g to *i*	$g+4h+i$
a to *i*	$a+4b+2c+4d+2e+4f+2g+4h+i$.

We have still to multiply by $\frac{1}{3}$ of the *single* interval to get the actual areas.

The sequence of the numerical multipliers is of importance. It will be observed that the multiplier of the first and last ordinates is always 1; of the first but one and last but one, always 4; and that, towards the middle, the 4 and 2 come alternately. There will be no difficulty in remembering this if we bear in mind that the numbers arise from the addition

```
141
  141      .
    141          .        . . .
      141    .            141
        . . .        .      141
14242424 and so on, ending with 4241
```

For example, let a series of offsets be 500, 526, 556, 588, 625, 667, 714, 769, 833, 909, 1000, and let the interval be 300. The work is done in a tabular form thus:

Parabolic method

Offsets	Multipliers	Products
500	1	500
526	4	2104
556	2	1112
588	4	2352
625	2	1250
667	4	2668
714	2	1428
769	4	3076
833	2	1666
909	4	3636
1000	1	1000
		20792
$\frac{1}{3}$ interval		100
Area		2079200

Trapezoidal Method

$\frac{1}{2}$	250
	526
	556
	588
	625
	667
	714
	769
	833
	909
$\frac{1}{2}$	500
	6937
interval	300
	2081100

The true value is 2079429·72. The parabolic method is generally more correct than the trapezoidal rule, and it is so in this instance. But when the curve is nearly at right angles to the base neither rule is accurate, and the parabolic method not much more so than the other.[1]

Symmetry of Curves.

There are certain curves which not unfrequently occur in practice, of which equal parts are known to lie above and below some right line, straight or inclined, so that, in fact, by taking the straight line instead of the curve, we shall add exactly as much as we cut off. For instance, see Fig. 24, in which the curve BC is simply the curve AB turned round B as a centre through two right angles. Any right line drawn through B, and meeting the right lines A*a*, C*c*, will then cut off from as much as it adds to the area *a*ABC*c*. There

[1] The parabolic method is usually known as Simpson's first rule. There is also a rule in which the intervals are grouped in threes instead of in pairs. The expression for three intervals, with four ordinates, *a*, *b*, *c*, *d*, is

$$\tfrac{3}{8}(a + 3b + 3c + d) \times \text{interval}.$$

If we take a series of these we may obtain the order of the multipliers by adding

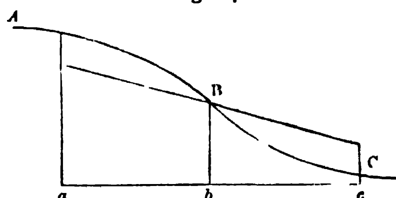
```
1331
   1331
      1331
         &c.
----------
1332332331 and so forth.
```

This is called Simpson's second rule. It is convenient when the number of *intervals* is a multiple of 3; but, for the same number of ordinates (as 7, 13, or 19), it is rather less exact than the first rule. The rules usually bear Simpson's name, although he was certainly not their inventor.

There are also more complicated rules which become of importance where it is difficult to get at the ordinates, and therefore where a few ordinates must be made the most of. But in general it is easier and better to take more ordinates than to use more complex rules.

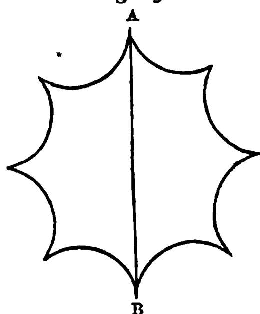
are certain other cases in which this will only happen with

Fig. 24.



some particular line. In such a case as this the area is at once known from its being the product of the base *ac* into the mean altitude *b* B.

Fig. 25.



There are also cases in which the curve is symmetrical to one or more lines, so that if the paper on which it is drawn be folded along that line, the two halves of the curve will fit. The curve is then divided into exact halves, and it is easy to see that there are many cases in which curves can be divided by mere symmetry into halves or quarters, or even into 6 or 8 equal parts.

Examples.

1. Find the area of the curve whose ordinates in inches are 26, 37, 50, 65, 82. Common interval 30 inches.

2. Find the area of the curve whose ordinates in inches are 74, 63, 50, 35, 18. Interval 30.

These areas together make up a rectangle, 100 × 120.

3. Find the area of a segment whose arc is a quarter of a circle by drawing and measuring offsets on its chord.

True area ·2854, radius being unity.

4. The offsets are ·1, ·14, ·173, ·20, ·224, ·245, ·265, ·283, ·3 in feet, and the interval is 1 inch. Find the area.

5. The true area of the curve from which the last example was taken is 20·8 square inches. In what proportion is the error of calculation increased by omitting half the ordinates and doubling the interval, both by Simpson's rule and by the trapezoidal rule?

6. The offsets are 1, 1·26, 1·44, 1·59, 1·71, 1·82, 1·9, 2, in centimètres, and the interval is 14 millimètres. Find the area, and reduce it to English inches.

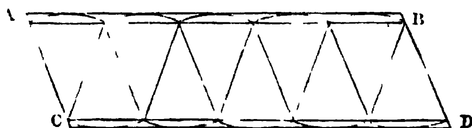
Circles.

If we take a circle and inscribe any regular polygon, and also circumscribe a polygon of the same number of sides, it is very clear, firstly, that the area of the circle will lie between the areas of these two polygons; and, secondly, that the circumference of the circle lies, in respect of length as well as of position, between the perimeters of the two polygons. Now let us suppose that we increase the number of sides of both polygons; say, by doubling them. Then it is clear that we shall have done two things: first, we shall have reduced the difference between the lengths of their perimeters; secondly, we shall have diminished the difference between their areas. If we could imagine these polygons with an infinite number of sides, it is clear that we should ultimately have their areas and perimeters both equal, and therefore also equal severally to those of the circle. In fact, they would not be distinguishable either from the circle or from one another.

The area of a regular polygon is the product of the perpendicular from its centre on the side, and of its semiperimeter, and it is easy to see that when the number of sides is very great indeed—so that practically the inscribed and circumscribed circles do not differ—this perpendicular becomes the radius of the circle. Hence the area of the circle is equal to the product of the radius and the semi-circumference. There is another way in which we may illustrate this. Consider the circle as cut up into a number of equal

sectors which are opened out and arranged as in Fig. 26. The circle is supposed to be cut in two, and the sectors of one semicircle fitted in so that all their corners lie on the parallel right lines AB and CD. This can clearly be done. Let us suppose also that we draw two other lines *ab* and *cd*, touching the arcs of the segments on the outside. It is plain that the area of the circle will lie between those of the parallelograms ABCD and *abcd*. Now the area of each of these parallelograms is the length AB × the perpendicular breadth. Let us, again, suppose that the number of sectors into which we have cut the circle is very large. The lines AB *ab* approach coincidence together, as do also the lines CD *cd*. The perpendicular distance between AB and CD

Fig. 26.



also tends to equality with the radius of the circle, while its semicircumference also merges in the line AB. Hence, if we suppose the number of sectors to be indefinitely increased, the area of the circle ultimately becomes,

$$\text{radius} \times \text{semicircumference}.$$

This is only another form of the illustration already given by means of the inscribed and circumscribed polygons.

If, as before, we call π the ratio of the semicircumference to the radius, or, what is the same thing, of the circumference to the diameter, we have—

$$\text{area} = \pi \times \text{square of radius}.$$

It follows that the area is equal to the square of the diameter × ·7854 nearly.

To find the radius from the area we must divide the area

by π, or multiply it by its reciprocal and take the square root.

Circular Annulus or Ring.—This is, of course, the difference of two circles. Let the radius of the larger circle be a, and of the smaller circle b, then the area of the annulus will be $\pi(a^2-b^2)$.

As a consequence of this, if a be the hypothenuse of a right-angled triangle, and b and c the other sides, the area of the annulus whose outer and inner radii are a and b is equal to the area of the circle whose radius is c.

This proposition is true not only of circular rings, but of any figure which is the difference of two similar figures.

Examples.

1. Find the area of a circle whose radius is 5 inches.

2. Find the area of a circle whose diameter is 7 inches.

3. Find the radius of a circle whose area is 10 sq. in.

4. Find the area of a circle whose circumference is 27 in.

5. Find the circumference of a circle whose area is a square mile.

6. Show arithmetically, by an example, that if a circle be inscribed in a square, the areas are in the same proportion as the perimeters.

7. A road 10 yards wide runs round a circular grass-plat 80 yards in diameter. What is the area of the road in a. r. p.?

8. The area of a ring is 9 square inches, and the external radius 3 inches. What is the internal radius?

9. The inner and outer circumferences of a ring are 15 and 17 inches. What is the circumference of a circle equal to its area?

10. The side of a regular pentagon is 2 inches, and another regular pentagon, whose side is 1 inch, is cut out of the middle of it. What is the area of the remainder?

11. What is the proportionate error of the following rough rule for finding the area of a circle?—Take $\frac{7}{9}$ of the square on the diameter and add 1 per cent.

Sectors of Circles.

It is evident that for sectors having the same radius the areas are proportionate to the angles or arcs. Hence a sector is the same portion of its whole circle as its angle is of four right angles, or of 360 degrees. It follows from this that sectors with the same angles but with different radii, vary as the square of the radius. We may thus express the area of a sector of 85° and of 3 inches radius, as $\frac{85}{360} \times 9 \times \pi = 6{\cdot}676$ square inches.

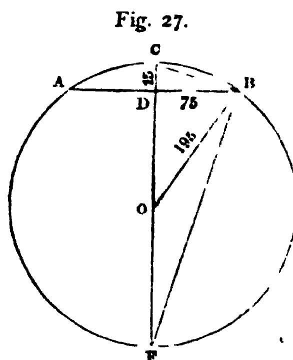
Segments of Circles.

The segment of a circle differs from a sector only by an isosceles triangle, as we may see by drawing lines from the centre of the circle to the extremities of the segment. Hence, if we know the following things : the radius of the circle, the angle of the segment, its chord, and the distance of its chord from the centre of the circle, we can find the area of the segment by common arithmetic. For instance, if the angle be 60°, the radius 10 inches, the length of the chord will be 10 inches, and the perpendicular on it from the centre will be $5 \times \sqrt{3} = 8{\cdot}660254$. Hence the area of the segment will be $100 \times \frac{\pi}{6} - 25 \times \sqrt{3} = 9{\cdot}0583$.

It is to be observed that of the four elements which have been given in this example, only two are chosen at pleasure, or are mathematically necessary to the question. When the radius and angle are given, the chord and its distance from the centre can be found by trigonometry. Moreover, we have not given even the arithmetic in its simplest form, for, instead of determining the triangle by means of the chord and the perpendicular, we might have determined it by means of one radius and the perpendicular on it from the other radius. This perpendicular is $5 \times \sqrt{3}$, and, in fact, the triangle is equilateral, so that in this particular case the

triangles not only come to the same, but are the same in detail. In this, as in most questions of mensuration, much depends upon the way the question is stated. It is usually a matter of geometry to reduce the question into its best shape for the application of arithmetic. We will take another example. A segmental arch is 150 feet span and 15 feet rise: required the area of the segment (Fig. 27). AB is the span of 150 feet, CD the spring or rise of 15. We must first find the radius of the circle geometrically. For this purpose I have simply to join CB, and draw BF at right angles to it, meeting CD produced in F. Then CF will be the diameter of the circle. If I want to determine DF arithmetically, I observe that by similar triangles

Fig. 27.



$$\text{CD} : \text{DB} :: \text{DB} : \text{DF}$$
$$\text{i.e. } 15 : 75 :: 75 : \text{DF}.$$

Therefore DF $= 375$ and CF $= 390$. The radius is therefore 195.

In order now to find the angle of the segment we must have recourse to trigonometrical tables. The angle of the segment is found to be 40° 30′ nearly. Hence the sector is

$$195 \times 195 \times \frac{40\frac{1}{2}}{360} \times \quad 13439 \text{ square feet.}$$

The perpendicular on the radius from the extremity of the other radius is

$$195 \times \cdot 649448.$$

Hence the triangle to be subtracted from the sector is $\frac{195 \times 195}{2} \times \cdot 649448 = 12347$ square feet; and the area of the segment is 1092 square feet.

It thus appears that the area of a circular segment cannot be obtained without trigonometry or the use of tables, unless more measurements are given than are strictly necessary to determine the question.

If tables of natural sines are at hand, the best expression for the area of a segment whose angle is given is

$$\text{sector} - \tfrac{1}{2} \text{ sine of angle.}$$

This supposes the radius to be unity. If it has any other value we must multiply the last term by the square of the radius.

Thus: given radius 2 inches, and angle 72°, also sine of 72° = ·95106,

$$\text{segment} = \tfrac{4}{5}\pi - 1{\cdot}90212 = {\cdot}61116 \text{ sq. in.}$$

Examples.

1. Find the area of a sector of 45° of a circle of 3 feet radius.

2. The area of a sector of 50° is 1 inch. Find the circumference of the circle.

3. The area of a certain sector of a circle of 3 inches radius is 10 square inches. What is the angle?

4. Find the area of a segment of 90° of a circle, whose radius is 4 inches.

5. Find the area of a segment of 60° of a circle, whose radius is 6 inches.

6. The sine of 54° to radius unity is ·809. What is the area of a segment of 54°, the radius of the circle being 13 inches?

7. The span and spring of an arch of 90° are 200 feet and 41·42 feet. What is the area of the segment?

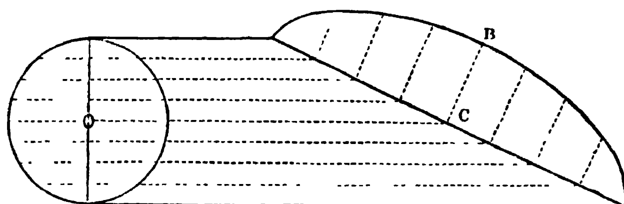
8. The spring of a circular arch of 240 feet span is 19·0056 feet and the angle is 36°. What is the area of the segment?

The Ellipse.

Before it is possible to measure a curve we must have some knowledge of the curve itself. Having acquired this, which is a matter of geometry, we have then to take that mode of considering the curve which is best suited to the application of arithmetic. For the purpose of measuring the area, an ellipse is best considered as a circular plate which has undergone uniform stretching in one direction, or contraction in the other. For instance, if a circular disc of metal be passed through a rolling mill, it will come out as an ellipse; not a very exact ellipse perhaps, because the rolling may not be done uniformly, and the edges may not be true. Another way is to draw a circle on the front edges of a closed book (which should be a pretty thick one), then slope the back of it, letting the leaves slide one over the other. Now if the front edges were a flat surface to begin with, and if the back be sloped so as still to keep them a plane surface, we shall again have an ellipse. Equivalent to these, but much neater, is to take a carefully turned cylinder of wood, and take an oblique section of it. For suppose we take this section, draw its longest diameter, and draw offsets at right angles to this, it will easily be seen that the length of these offsets is independent of the obliquity of the cut, but they are further apart or nearer together, according as the cut is more or less oblique. This may be easily verified if the cylinder is made up of planks glued together and then put in the lathe lengthwise (see Fig. 28). It is easy to see that the area of this ellipse will simply be the area of the circle from which it is obtained, multiplied by the ratio of stretching or contracting. This ratio of stretching or contracting may be inferred by considering simply the stretching from the shortest diameter to the longest, or the contracting from the longest to the shortest. If we call these two diameters A and B, A being

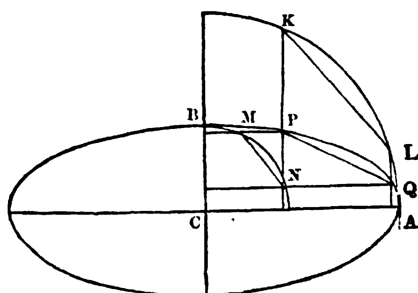
greater than B, we may either consider it as a circle, diameter B, which has been stretched in one direction only, in the

Fig. 28.



ratio of A : B, or a circle, diameter A, contracted in one direction only, from A to B. Hence the area of the ellipse is $\frac{\pi}{4}$ A B, while the area of the larger circle is $\frac{\pi}{4}$. A. A $= \frac{\pi}{4}$. A^2, and of the smaller circle, $\frac{\pi}{4}$. B^2. This is not only true of the whole ellipse, but of any segment (see Fig. 29). The segments in the ellipse bear the same ratio to the segment in the circle as the whole circle does to the whole ellipse. Thus if A C and C B be in the ratio of 7 : 5, the elliptic segment P Q is $\frac{7}{5}$ths of the circular segment M N, or $\frac{5}{7}$ths of the circular segment K L.

Fig. 29.



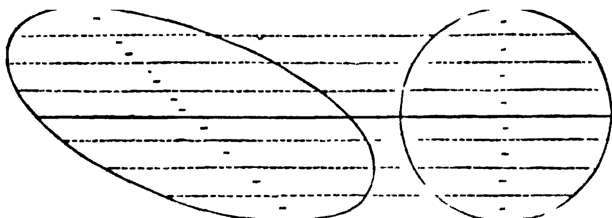
Let it be required to find the area of an ellipse whose greatest and least diameters are 7 and 5 feet. The area will be $7 \times 5 \times {\cdot}7854 = 70 \times {\cdot}3927 = 27{\cdot}489$ square feet. Geometrical readiness has of course much to do with mensuration. Suppose, for instance, it were desired to measure the segment cut off from the ellipse in the last example by

the chord which joins the extremities of the greatest and least diameters. Instead of finding the value of the similar segment in a circle of 5 feet diameter, and increasing its area in the ratio of 7 : 5, it is just as easy to subtract at once the area of the triangle A B C from the quarter ellipse. This gives

$$6{\cdot}872 - \tfrac{35}{8} = 6{\cdot}872 - 4{\cdot}375 = 2{\cdot}497 \text{ square feet.}$$

Another useful mode of looking at the ellipse is to consider it as obtained from another ellipse, which has been cut into slips, and these have been made to slide one upon another, as in the diagram.

Fig. 30.



Evidently, if the number of slips be indefinitely large, the area will remain unaltered.[1] It follows that all ellipses lying between the same parallels have their areas proportional to the diameters also parallel. Accordingly the areas of the ellipse and circle in Fig. 30 are in the same ratio to one another as the chords made in each by any one of the horizontal lines. If, therefore, we take any diameter of an ellipse (which call $2m$), and draw a tangent parallel to that diameter (call the distance between the two lines n), then the area of the ellipse will be $\pi m n$.

[1] According to the point of view from which we start, this may be regarded either as a proof or a consequence of the geometrical theorem, that the variable parallelogram circumscribing a fixed ellipse, and having its sides parallel to conjugate diameters, is of constant area.

Examples.

1. Find the area of an ellipse whose axes are 3 inches and 2 inches.

2. Find the area of an ellipse whose axes are 2 miles and 1 mile.

3. A pair of parallel tangents to an ellipse are 3 feet apart, and the diameter parallel to them is $4\frac{1}{2}$ feet long. What is the area of the ellipse?

4. What is the area of the segment of an ellipse made by joining the extremities of the axes?

5. Show that the area of an annulus is equal to that of an ellipse whose axes are the sum and difference of the external and internal diameters of the annulus.

6. The area of an ellipse is 25 sq. inches, and one of its axes is 8 inches long. What is the other?

7. What is the radius of a circle whose area is the same as that of an ellipse whose axes are 24 and 36 inches respectively?

8. A parallelogram with four equal sides has an ellipse inscribed in it, so that the axes lie upon the diagonals. These diagonals are 6 inches and 14 inches. What is the area of the ellipse?

CHAPTER XX.

MEASUREMENT OF CAPACITY OR VOLUME.

The capacity or volume of a rectangular block in cubic units is the continued product of the length, breadth, and depth in linear units. This is obvious enough when the length, breadth, and depth can be expressed each as a whole number of these units or of any smaller units, for we may first cut the block into flat plates one unit thick, we may then cut these plates into square rods one unit broad and one unit thick; and the number of these square rods will be units of breadth multiplied by units of thickness. When again we cut these into cubic units, it is clear that the total

number is as stated. When the number is fractional the same thing holds, although not so obviously; but it can be proved by reasoning similar to that used for rectangles.

It is on this principle that depends the measurement of such things as iron plates or sheet lead, in which the thickness has to be taken into account. When, however, the block is not rectangular, but still has its opposite sides parallel, its volume is measured by the product of the altitude in linear units into the area of the base in square units. This can be proved, as in the case of the parallelogram, for the case in which the obliquity is single, that is, where two sets of faces are rectangular; and a second or third application of the principle will prove it in the cases in which the block is oblique every way. But the more instructive way of considering the question is as follows. Let us imagine a rectangular block divided into layers like a ream of paper in half sheets or a pack of cards. It is evident that we shall not alter the quantity of paper or cardboard by sliding the cards one upon another: nor shall we alter the thickness of the pack. Let us, therefore, by pressing against the edges, slide the cards so as to make their edges form a rough but practically a plane surface. Neglecting the roughness of the edges, we have now got a parallelepiped, of which the volume remains unaltered from the square block. Its volume is therefore still the area of the base × the altitude taken perpendicularly to the base. The case of treble obliquity is easily managed by assuming the cards to be lozenges instead of rectangles. The proof is the same as has already been given for the area of the parallelogram, namely, that the outlying and inlying solids tend to coincidence as the steps are taken thinner and more numerous.

The principle which we have stated for the measurement of a rectangular solid, namely, that its solid content is equal to the product of the base into the altitude, evidently applies to all solids whose cross section does not vary, whatever may be the shape of that cross section, regular or irregular,

such as, for example, a strip of lead which has been forced through a die, like casement lead, or an ordinary moulding which has been shot with a moulding plane. It is clear that we shall get the solid contents by multiplying the transverse area in square units by the length in linear units, for this is nothing more than saying that the rod or moulding is of the same bulk as a rectangular rod of the same transverse section and length. This leads us at once to the conclusion that any right cylinder or right prism has its volume measured by the product of the base into the altitude. Thus, to take the case of a pipe 2 feet in diameter internally. Its section will be 3·1416 square feet, and therefore its internal capacity will be 3·1416 cubic feet *per foot run*, as it is called. The very expression 'per foot run' implies that the volume varies in the same ratio as the length.

Let us next consider an oblique cylinder, the top and bottom being circular and horizontal, but not exactly one over the other. Let us take a right cylinder with the same base, and of the same vertical height, and consider it divided into horizontal layers, as if it had been punched out of a pack of cards or a ream of paper. We may slide the cards or sheets of paper one upon another until we change the right cylinder into the oblique cylinder. It is therefore evident that the volume of an oblique cylinder is measured by the product of its altitude into its base, the altitude being measured perpendicularly to the base. It is not even necessary for this that the shape of the base should be restricted in any way—to a circle or any other form. The only restriction is that the parallel section should be uniform.

Measurement of Pyramids.

By a pyramid I mean a solid having a polygonal base and a point not in the plane of the base. This point, called the vertex, is joined by straight edges to every angle of the

polygon on the base. All sections parallel to the base are similar.

The measurement of the volume of pyramids depends upon the following geometrical principles :

1. Pyramids on the same base and of equal altitude are equal to one another. This is seen by cutting the pyramid into thin slices parallel to the base, and sliding them one over another so as to form another pyramid.

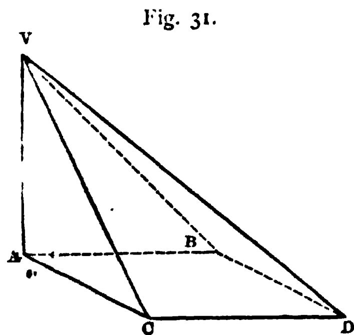
2. Pyramids on equal bases and of equal altitude are equal. For if, in what went before, we substitute for each slice an equal slice, the sum or volume will not be altered.

3. Pyramids of equal altitude are in the same ratio as their bases. For if we alter every one of the slices of which the pyramid is composed, we shall alter the volume of the pyramid in the same ratio.

This does not enable us to find the actual volume of any pyramid; but it so happens that there is one particular pyramid of which we can find the volume in terms of the base and altitude, and that is a pyramid whose base is a square, whose altitude is equal to the side of the square, and whose vertex is directly over one corner of the square.

If we take three such pyramids we can fit them accurately together into a cube, of which A B C D will be one face and V A one edge. Hence in this case the volume of the pyramid is $\frac{1}{3}$ base × altitude. Bearing in mind the last of the three principles just laid down, we see that this proposition must be true of any pyramid whatever. Therefore to find the volume of any pyramid in cubic units, take $\frac{1}{3}$rd of the product of the altitude (in linear units) into the base (in square units).

Fig. 31.



The same rule applies, by virtue of the second of these

principles, if we substitute for the polygon ABCD any curved figure, the solid being marked out in space by a line passing always through the vertex and travelling round the edge of the figure. It is, in fact, true of a cone, whether right or oblique, and whether the base be circular or not, that the solid content is one-third of the base into the altitude.

Any body bounded by plane surfaces can have its volume measured by splitting it into pyramids. This is done by taking any convenient point and drawing lines from that point to all its corners. Care must of course be taken to see whether each pyramid is additive or subtractive. For instance, if in the triangular pyramid ABCD we take a point O inside the pyramid, we shall in effect split it up into the sum of the four pyramids OABC, OBCD, OCAD, and OABD. If, on the other hand, we take the point outside, so that the pyramid OABC wholly contains the pyramid ABCD, the volume of the pyramid ABCD will be the difference between the pyramid OABC, and the sum of the pyramids OBCD, OCAD, and OABD, that is, these last three pyramids must be taken subtractively (Fig. 33). Each pyramid may then be separately measured by letting fall a perpendicular from the point O on the base. The illustration which I have just given was intended simply to show how subtractive quantities came in. The pyramid ABCD might of course be measured by finding the area of any one of the sides, and the altitude taken with that side for base.

Fig. 32.

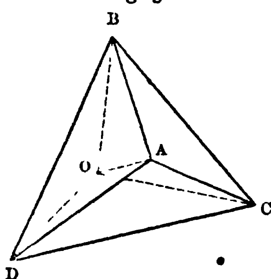
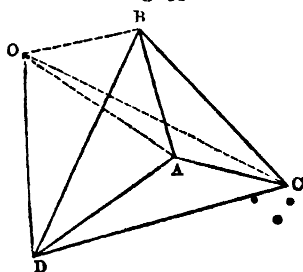


Fig. 33.



Examples.

1. The cross section of a pipe is 1·5 square feet. How many gallons of water does it contain per yard run?

2. What is the cubic content of a cylinder 108 inches diameter and 10 feet long?

3. An elliptic cylinder is 12 feet long. The axes are 8 feet and 6 feet respectively. How much water will it contain?

4. What is the volume of a pyramid whose base is a rectangle 13 × 14 feet, and whose height is 18 feet.

5. A right-angled triangle is lying horizontally. Lines are drawn to the angles from a point 17 feet above the plane of the triangle. Supposing the sides about the right angle to be 9 and 14 feet, what is the volume of the pyramid?

6. A gallon measure is a cylinder 6 inches high. What is its diameter?

7. Another gallon measure is a cylinder of the same height as breadth. What is the height?

8. A cone is 5 inches high, and the radius of its base is 3 inches. How many pounds of mercury, sp. gr. 13·5, will it contain?

9. The cylinder of a hydraulic press is of cast iron, sp. gr. 7·1, external diameter 18 inches, internal 7 inches, and 12 feet long. What is its weight?

10. What will be the mechanical work of pumping out a well 5 feet in diameter and 200 feet deep, the water being 40 feet from the top?

11. A steam engine has two 84-inch cylinders, 4 feet stroke and 30 strokes a minute. The mean pressure is 28 lbs. What is the indicated horse power?

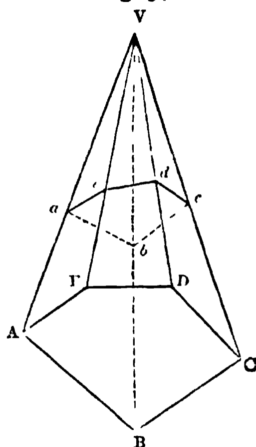
12. A single-acting pump, 14 inches diameter and 12 feet stroke, is doing 8·5 strokes a minute, and raising water from a depth of 240 yards. What is the effective horse power?

Of Truncated Pyramids and Cones.

These are best considered as the difference between two cones or two pyramids; thus the truncated pyramid *a b c d e*

A B C D E is simply the difference between the two pyramids V A B C D E V *a b c d e*, and as the volume of each pyramid is simply one-third of the base × altitude it is easily found when these are known. This is equally the case whether the plane sections *a b c d e* and A B C D E are parallel or not. The determination of the altitudes is a question of geometry or trigonometry, or of actual measurement.

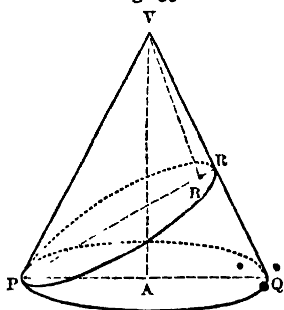
Fig. 34.



When the base and top of the truncated cone or pyramid are parallel, it is very easy to supply the portion required to complete it, by the consideration that the portions of the indefinite cones or pyramids cut off by the two planes are similar. The heights will then be in the same proportion as the diameters or sides, and as the square roots of the areas.

The conic ungula is simply a particular case of the truncated cone in which the sections are not parallel. The solid P Q R (shaped like a horse hoof) is called the ungula. It is the difference between the two cones V P R and V P Q, whose altitudes are V A and V B respectively. To measure this we require means of ascertaining the areas P Q and P R, and the heights V A and V B.

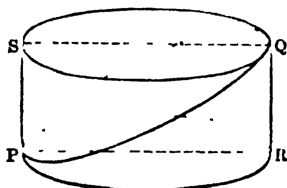
Fig. 35.



Cylindric ungula.—This is evidently one-half a cylinder whose height is that of the ungula. The ungula P Q R is half the cylinder P S Q R, whether the cylinder be right or inclined. It is not even necessary that the base should be circular;

the base may be any oval whatever which is symmetrical to two diameters at right angles to one another.

Fig. 36.



Examples.

1. A cylindrical jar 5 inches in diameter and full of water, is tilted until $2\frac{1}{2}$ lbs. of water has run out. How far is the surface of the water below the upper edge of the cylinder?

2. A conical glass, 3 inches diameter at the top and 4 inches deep, is half filled with water. How much does it contain, and how high does it come in the glass?

3. A pyramid standing on a rectangle, 80 × 60 feet and 50 feet high, is cut off at a height of 20 feet from the ground. What is the cubic content of the frustum?

4. A round spar 60 feet long tapers regularly from 10 inches diameter at the top to 24 inches at the butt. How many cubic feet of timber does it contain?

5. The unfinished spire of a cathedral tower is octagonal, and measures 80 feet round at the base. At the height of 55 feet above the base it measures 30 feet round, and the builders have there stopped. How many cubic feet of stone does it contain?

6. An elliptic cone is cut by two planes, distant respectively 3 and 5 inches from the vertex, and giving circular sections of 3 and 4 inch diameters. What is the volume included between them?

7. A hollow cone is 10 inches internal diameter at the base, and 10 inches high on the inside, and 12 inches ex-

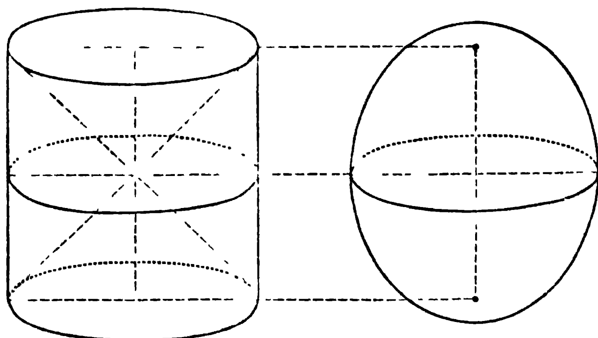
ternal diameter. What is its weight, supposing it to be made of brass uniformly thick, and sp. gr. 8?

8. Two right cones have the same circular base of 8 feet diameter, and one of them is 10 feet high and the other 12, both vertices being turned the same way. How many gallons will the space between them contain?

Volume of the Sphere.

The connection between the sphere and the cylinder is best seen by taking a cylinder of the same diameter as the sphere, and of a height equal to its diameter. Then scoop out of the cylinder two equal right cones each having the base of the cylinder for its base, and meeting in the middle point of the cylinder. Put it alongside of the sphere, as shown in the drawing (Fig. 37). It is very easily shown by

Fig 37.



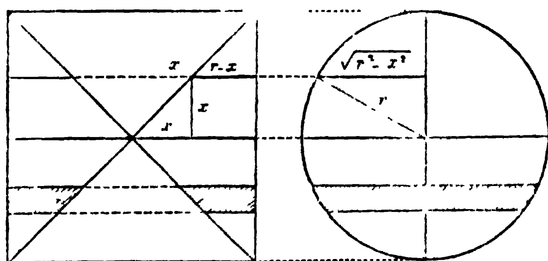
geometry [1] that any horizontal section of the sphere and of

[1] Let the radius of the sphere be r, and the height of the slice from the centre x. Then the radius of the small circle of the sphere will be $\sqrt{r^2-x^2}$ and its area $= \pi \times (r^2-x^2)$. The section of the other solid will be a ring which is the difference of two circles—one of them equal to the base of the cylinder, or πr^2. Of the other it is clear from the shape of the cone that the radius will be equal to the height x. Its area will therefore be πx^2 and the area of the ring will be $\pi \times (r^2-x^2)$. This is the same as that of the small circle.

this solid taken at the same height are equal, therefore the entire solids are equal. But the cones which have been scooped out are each $\frac{1}{3}$rd of the cylinders of the same height, and taken together they form $\frac{1}{3}$rd of the cylinder. Therefore the volume of the sphere is $\frac{2}{3}$rds of the cylinder. The above construction has the advantage of showing how finding the volume of any parallel slice of a sphere may be reduced to that of finding the volume of a truncated cone. The volume of the cylinder is, as already stated, the product of the altitude and of the area of the base. Now if we call the radius of the sphere A, the radius of the cylinder is the same, the area of the base is πA^2, and the altitude is $2A$. The volume of the cylinder is $2\pi A^3$. The whole volume of the sphere is therefore $\frac{4}{3}\pi A^3$.

It is also evident that any horizontal slice, however thick, of the sphere, is equal to the corresponding hollow slice of the cylinder, with a truncated cone cut out of it. See Fig. 38, which is an elevation of Fig. 37. The solid content of the shaded slice is the same in both parts of Fig. 38.

Fig. 38.



The segment of a sphere thus presents no difficulty, being simply a particular case of the zone, in which the upper circle vanishes.

For example, required the solid contents of a sphere of 12 inches diameter, and of a segment of that sphere whose height is 3 inches.

The content of the sphere in feet is $\frac{4}{3}\pi \times \frac{1}{8} = \frac{\pi}{6} = \cdot 5236$, or $904\cdot 6$ cubic inches. For the segment, the corresponding portion of the cylinder is $\frac{1}{4} \times \frac{1}{4} \times \frac{1}{2}\pi = \frac{1}{16}\pi$, and out of this we have to cut a truncated cone, equal to the difference of two cones, each with the radius of base equal to their height, one $\frac{1}{4}$ foot high, and the other $\frac{1}{2}$ a foot. The content of the truncated cone is therefore $\frac{1}{3}\pi \times \frac{1}{8} - \frac{1}{3}\pi \times \frac{1}{64} = \frac{7\pi}{192}$. Subtracting this from the cylinder $\frac{\pi}{16}$ we have for the segment $\frac{\pi}{16} - \frac{7\pi}{192} = \frac{5\pi}{192} = \cdot 08018$ cubic feet $= 138\cdot 5$ cubic inches.

In this case we suppose the depth of the segment and the radius of the sphere to be both known. If other data are given, we must find these from them, either by geometry or trigonometry.

The Spheroid and the Ellipsoid.

A spheroid is a solid made by spinning an ellipse about one of its principal diameters. If about the shortest diameter, it is called an oblate or flattened spheroid; if about the longest diameter, a prolate or elongated spheroid. Now in exactly the same manner as we considered the ellipse to be a circle which had undergone either compression or stretching, so we may consider an oblate spheroid to be a sphere which has undergone uniform compression in the direction of its axis, and the prolate spheroid to be a sphere which has undergone uniform expansion in the direction of its axis, and in that direction only. In this way, if a and b be the longest and shortest semidiameters respectively, the volume of the oblate spheroid will be $\frac{4}{3}\pi a^2 b$, and of the prolate spheroid $\frac{4}{3}\pi ab^2$. For these are only the expression for the volume of the sphere diminished or increased in the ratio of contraction or expansion. This consideration enables us to deduce the volume of any zone of the ellipsoid bounded by planes perpendicular to the axis of the spheroid.

Instead of regarding the oblate spheroid as a sphere which has undergone compression in one dimension, we

may regard it as a sphere which has undergone equal expansion in two directions at right angles to one another. If, however, instead of these expansions being equal, they are unequal, we shall get a figure which is called an ellipsoid. Suppose we start with a sphere whose radius is c, and we stretch or contract it one way in the ratio of $c : b$, and we stretch or contract it the other way (at right angles to the former) in the ratio of $c : a$, it is clear that we alter the volume from $\frac{4}{3}\pi c^3$ to $\frac{4}{3}\pi abc$.

Examples.

1. What is the cubical content of a sphere 7 inches in diameter?

2. What is the radius of a sphere containing a gallon?

3. What is the area of the great circle of a sphere containing a cubic foot?

4. The area of a great circle of a sphere measures a square foot. What is the cubic content of the sphere?

5. What is the weight of an iron shell whose internal and external diameters are 7 and 13 inches respectively, and what weight of gunpowder will it hold? Specific gravity of iron 7·5, and of gunpowder 1·8.

6. The earth's mean equatorial axis is 12755 kilomètres and its polar axis 12712·2. How many cubic mètres does it measure?

7. Assuming the atmosphere to extend in all directions 50 kilomètres above the earth, how many cubic mètres does it measure?

8. What is the cubic content of an ellipsoid whose principal diameters are 3, 6, and 12 feet?

9. A hemi-spheroidal cup measures 10 inches across the top, and is 8 inches deep. How many gallons does it hold?

10. Supposing it is filled to a depth of 3 inches, how much will it hold then?

11. What portion of a sphere 12 inches in diameter is contained between two parallel plates distant 2 and 4 inches from the centre, and on the same side of it?

12. By how many gallons does the cubic capacity of an ellipsoid whose principal semidiameters are 4, 5, and 8 inches, exceed that of a sphere of 5 inches radius?

CHAPTER XXI.

AREAS OF CURVED SURFACES.

The only curved surfaces which can usefully find a place in an elementary treatise on Mensuration are cylinders, right cones, and spheres. The surface of a cylinder consists in general of three distinct portions—the two ends and the curved surface. The two ends fall within the rules for measuring plane areas, and the curved surface alone remains to be considered.

Right cylinder.—If we imagine the cylinder to be a hollow one, consisting of a very thin sheet, we may imagine the curved surface split up one side, and then laid flat. Its area is then seen to be rectangular, one side being the altitude, and the other the circumference. Calling the altitude A and the radius B, the area will be $2\pi AB$.

The area of each end will be πB^2, the total for the whole cylinder will therefore be $2\pi B \times (A + B)$.

Next consider a circular cylinder, of indefinite length, cut by two parallel planes oblique to the axis. By 'circular' I mean a cylinder such that a section at right angles to its axis gives a circle. Let A be the radius of this circle, and B the length of the axis comprised between the two planes of section. Then the curved surface will be, as before, $2\pi AB$.

These oblique sections will be ellipses, and their centres will be marked by the axis. Now, through these centres draw two planes perpendicular to the axis. These will meet the cylinder in circles, and it is evident that the sort of wedge between the direct and oblique sections will be, in all respects, the same above and below the plane of

section, and that the curved surface cut off on one side will be equal to the curved surface cut off on the other, so that, substituting an oblique section for a direct section through the same point of the axis, alters neither the surface nor the volume. This is still the case if we do so at both ends of the cylinder, provided only the two oblique planes do not intersect within the cylinder. As we have supposed them to be parallel they do not intersect at all, and therefore the oblique belt has the same area as the direct belt of the same length measured along the cylinder.

It follows from what we have said that if a circular cylinder be cut by any two planes which do not intersect within the cylinder, the area of the curved surface comprised between them will be the product of the circumference of the cylinder into the length of the axis intercepted between the two planes.

The particular case in which the sections are parallel can be derived from the case in which the section is direct, by supposing the surface to be cut up into strips parallel to the axis of the cylinder, and these strips to be slid along the cylinder until their ends come against the line of oblique section. If the two planes of section intersect within the cylinder there will be two wedges, and it is the difference between the curved surfaces of these wedges which will be given by the product of the circumference into the length of axis intercepted. This is not of much practical use, but it is simply added as a caution. The measurement of the surface of the wedges singly requires the use of trigonometrical tables.

Cylinders not Circular.

The curved surface of any belt comprised between two direct sections, or two oblique parallel sections, will, in the same way, be found to be the length measured along the cylinder, multiplied by the girth, this girth being the shortest girth measured directly round the cylinder, but following the

concavities, if any there be. In practice this girth is obtained by actual measurement, or by dividing the curve into short lengths, and setting off each length separately along a right line, as if it were straight. The only error thus introduced is the difference between a small arc and its chord, which may be disregarded.

Examples.

1. What is the curved surface of a right cylinder 3 inches in diameter, and 4 inches high? What is its entire surface?

2. A cylinder, 1 foot in diameter, is cut by two planes which meet the axis in points 2 feet apart. What are the intercepted curved area and contents?

3. Supposing the planes to make an angle of 45° with the axis, what is the entire surface? Why is it necessary to add some such supposition?

4. A right elliptic cylinder has a girth of 27 inches and a height of 6 inches, what is its curved surface?

5. Find the weight of a Lancashire boiler (cylindrical, 6 feet in diameter, with two circular flues 2 feet in diameter), 40 feet long, of $\frac{1}{2}$-inch iron, allowing 10 per cent. for lap-joints, stays, rivets, &c.

6. What is the cost per foot run of copper tubing 4 inches in diameter, and $\frac{1}{8}$ inch thick, at $7\frac{1}{2}d.$ per lb., specific gravity 8·8?

The Right Cone.

In the right cone every point on the circumference of the base is equidistant from the vertex and the straight line joining it to the vertex lies wholly upon the surface of the cone. If, therefore, we slit up the curved surface, we can spread it out flat, and it will evidently form a sector of a circle whose radius is the slant side of the cone. Moreover, since the circumference of the base has been spread out into the arc of the sector, it is clear that this arc will be the

same fraction of the circle that the radius of the base is of the slant side, because the circumferences of circles are in the same ratio as their radii. If S be the slant side, the circle drawn with that as radius will have πS^2 for its area, and the area of the sector made by developing the curved surface of the cone will be πSB. This is equal to the curved surface of a cylinder of the same height as the slant side of the cone, and of half the radius of base; or to the area of an ellipse one of whose semi-axes is the slant side, and the other the radius of the base. For instance, let the slant side of a cone be 5 inches and the radius of the base 2 inches. The area of a circle with 5 inches radius is 78·54 inches, and the sector into which the curved surface of the cone can be developed is $\frac{2}{5}$ of this or 31·416. Further, the area of the base will be 12·5664; therefore the complete surface of the cone, base and all, will be 44·0824 square inches.

It does not always happen that the slant side and radius of base are both explicitly given. We then have virtually, if not actually, to find these: if the angle be given we shall generally need the trigonometrical tables, but if the altitude be given we can get at the result simply by right-angled triangles. For, let the height be H and the radius of base B, the square of the slant side will be $H^2 + B^2$, and the area will be $\pi B \times \sqrt{H^2 + B^2}$.

Truncated Cones.

These may be regarded as the difference between two cones with the same angle at vertex. It may be easily shown, either geometrically or algebraically, that the curved surface of a truncated cone is equal to the curved surface of a cylinder of the same height as the slant side of the truncated cone, and with a diameter equal to the sum of the top and bottom radii. For instance, let the slant side of the truncated cone be 7 inches, the radius at top 8 inches,

and the radius at bottom 12 inches. The radius of the cylinder of equivalent surface will be 10 inches, and its girth 62·832 inches, and the curved surface both of cylinder and truncated cone, will be $7 \times 62·832 = 439·824$ sq. in. To verify this, I remark that this truncated cone may be regarded as the difference between two cones fitting into one another whose slant sides are severally 14 and 21 inches, and the radii of their base 8 and 12 inches respectively. The curved surface of the larger cone is $12 \times 21 \times \pi$; the surface of the smaller cone is $8 \times 14 \times \pi$. Their difference is, therefore, 140π, or, as before, 439·824 square inches.

The measurement of the surface of cones which are not right cones must be made by actually developing them and measuring them as plane surfaces.

Examples.

1. Find the total surface of a cone whose slant side is 20 feet and radius of base 10 feet.

2. Find the curved surface of a cone whose height is 15 inches and radius of base 20 inches.

3. Find the curved surface of a truncated cone whose top and bottom diameters are 7 and 10 inches and whose slant side is 5 inches.

4. Find the total surface of a truncated cone whose top and bottom radii are 4 and 9 inches and whose altitude is 12 inches.

5. Find the weight of a boiling pan in the form of a truncated cone, bottom diameter 20 inches, top diameter 40 inches, with a flange of 2 inches all round the top edge; depth 20 inches; no lid; $\frac{5}{16}$ iron; specific gravity 7·7.

6. Find the radius of a circle which shall be equal to the entire surface of a cone 20 inches high and radius of base $7\frac{1}{2}$ inches.

7.* What will be the angle of slope of a cone which developes into a semicircle?

Surface of the Sphere.

If any polyhedron be circumscribed to a sphere the surfaces of the two are in the same ratio as their volumes. This is not only true of the entire solids, but of the portions of them comprised within the surface of any cone or pyramid drawn from the centre of the sphere as vertex. For if we consider a very acute cone or pyramid, the small portion of spherical surface which it cuts out may be considered as practically flat. The volume of the portion of the cone or pyramid comprised within the sphere will therefore be the product of $\frac{1}{3}$rd of the altitude into the spherical base. Again, the volume of the cone or pyramid comprised within the polyhedron will be the base on the plane of the polyhedron multiplied by $\frac{1}{3}$rd of the altitude. The altitude is evidently the radius of the sphere; hence the volumes of the cones or pyramids will be in the same ratio as their bases. Now, if we consider the whole sphere as cut up into a very fine network by very acute cones or pyramids from the centre, we have simply to add them all together to get the proposition.[1]

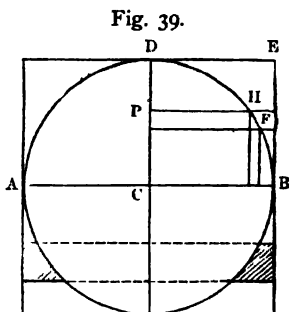
Next, consider a cube with its inscribed sphere, and suppose the radius of the sphere to be A, the volume of the sphere will be $\frac{4}{3}\pi A^3$; the volume of the cube will be $8A^3$, and the surface of the cube will be $24A^2$. Now, in order that the volumes may be proportional to the surfaces, the surface of the sphere must be $4\pi A^2$; it is therefore equivalent to four times the area of a great circle, or to the curved surface of the circumscribing cylinder.

There is another way in which we may get directly at the comparison between the sphere and cylinder. Consider

[1] Moreover portions of any conical or cylindrical surfaces touching the sphere in arcs of circles may be substituted for the planes of the polyhedron. The proof given applies to this. It follows that the ratio of the sphere to its circumscribing cylinder or cone is the same for volume as for surface.

the sphere and its circumscribed cylinder, both cut by a pair of planes at right angles to the axis of the cylinder, and therefore parallel to one another. I say that the zones which they trace out on the sphere and cylinder are of equal surface. For, suppose the interval to be very small (Fig. 39), so that HF may be considered for practical purposes as a straight line. Then it can be shown geometrically that the obliquity of the spherical zone increases its surface from what it would be if it were parallel to the axis, in exactly the same ratio as its horizontal radius PH is less than the radius of the sphere CB.

Fig. 39.



This is, in point of fact, regarding the sphere as made up of an indefinite number of conical belts, each excessively narrow, and each equal to the corresponding ring on the cylinder. Adding these together we arrive at the proposition that any zone of the sphere has the same surface as the corresponding zone of the cylinder, and that the whole surface of the sphere is equal to the curved surface of the cylinder. In other words, the curved surfaces inside and outside the shaded ring are equal to one another.

As a spherical segment is simply a zone of which one radius has vanished, it falls within the rule, and its curved surface is seen to be its altitude, multiplied by the circumference of a great circle. For example, let the radius of a sphere be 1 foot and the thickness of a segment 1 inch, its area in square feet will be $\frac{\pi}{6}$ or ·5236 square feet.

There is only one other way in which a sphere can be cut up so as to come within elementary mensuration, and that is by means of meridians, which are a series of great circles all passing through the same pair of points, called poles,

opposite to one another on the sphere. Their distance from one another is measured either by the arc which they cut off on a circle drawn from one of the poles as centre, or by the angle at which they intersect at the pole. It scarcely needs demonstration to see that the surface comprised between any two of the semicircles is the same portion of the whole surface of the sphere as the angle between them is of four right angles, or of 360°. Thus, considering the earth spherical, the area comprised between the meridian of Greenwich and that of 40° west, is $\frac{40}{360}$ths or $\frac{1}{9}$th of the whole surface of the earth.

Examples.

1. Find the surface of a sphere whose radius is 6 inches.

2. What is the weight of a hemispherical copper sugar pan, mean radius 2 feet, and $\frac{3}{16}$th thick, specific gravity 8·9?

3. A segment of a sphere of 5 feet in diameter is 3 inches deep: what is the surface of the segment?

4. Show that the surface of any zone of a sphere is equal to that of a circle whose radius is a geometrical mean between the diameter of the sphere and the height of the zone.

5. Hence find a circle whose area is equivalent to the segment mentioned in Example 3.

6. Assuming the mean amount of watery vapour in the atmosphere to be 3 per cent. of it, and the mean atmospheric pressure 14·7 lbs. per square inch, what will be the total quantity so suspended, taking the earth as a sphere with a mean radius of 20,888,800 feet?

7. 20 lbs. of copper are planished into a hollow hemisphere of 6 inches radius; specific gravity 8·9. What will be the thickness?

8. 20 lbs. of copper are planished into a hollow spherical segment $\frac{1}{8}$ in. thick, and whose depth is one quarter the radius of the sphere from which it is struck. What is that radius, and what is the diameter of the segment?

CHAPTER XXII.

THE PROPERTIES OF PAPPUS.

THESE principles, sometimes called the laws of Guldinus, enable us, under certain conditions, to find the volumes and surfaces of solids of revolution; that is to say, of solids which can be turned in the lathe. They are as follows: If a closed plane figure be made to revolve round an axis outside it and in its own plane—

1. The area of the surface will be the product of the perimeter of the curve multiplied by the length of the path of the centre of gravity of the curve.

2. The volume will be the product of the area of the curve into the length of the path of the centre of gravity of the area.

Thus, suppose a square of 1 inch to revolve round an axis parallel to one of its sides and 3 inches distant. The centre of gravity of the area and perimeter of the square will coincide and will be $3\frac{1}{2}$ inches distant from the axis. Hence the path of the centre of gravity will be 7π. Since the area of the square is 1 square inch, the volume swept out will be 7π or 21·9912 cubic inches. This may easily be verified by reflecting that the solid in question is a cylinder 8 inches diameter, and 1 inch high, with another cylinder 6 inches diameter and 1 inch high cut out of the middle of it. Hence the volume is $16\pi - 9\pi = 7\pi$ as before. Again, for the surface, the path of the centre of gravity is as before 7π, and the perimeter of the square is 4 inches. The surface will therefore be 28π or 87·9648 square inches. The student will find no difficulty in obtaining a verification of this by taking the outer and inner curved surfaces, and the top and bottom separately, and adding them.

The useful application of these rules, however, is to cases in which it is easy to see what the centre of gravity is, but in which it is not easy to measure the surface in any other manner. Take, for example, the case of an anchor ring made by spinning a circle 2 inches in diameter round an axis 5 inches from the centre. The centre of gravity both of area and circumference is here the centre of the circle. Its area is π inches, and its circumference 2π inches, while the path of the centre of gravity is 10π. The volume is

therefore $10\pi \times \pi$ cubic inches, and the area double that number of square inches, that is, the volume is 98·7 cubic inches, and the area 197·4 square inches nearly.

It is necessary, in the practical application of these rules, that the curved surface should be swept out once, and once only; otherwise, if there be any overlapping, the geometry and the arithmetic will not directly correspond. For instance, if a complete circle 2 in. in diameter be made to revolve round an axis $\frac{1}{2}$ in. from its centre, the two segments will describe different spindles, and the product of the area of the circle into the length of the path traced by its centre will not give the volume of either, but the difference between the volumes of the two spindles. To find the volume of either separately we must multiply the area of the segment into the path of its own centre of gravity. This is of course less simple, inasmuch as it requires that we should know the area of the segment, and the position of its centre of gravity.

Examples.

1. Find the volume of the solid swept out by an ellipse whose axes are 8 and 16 feet, revolving round an axis in its own plane, and 10 feet from its centre.

2. An equilateral triangle, 1 inch a side, revolves round an axis 1 inch distant from its centre of gravity. Find the volume and area of the solid swept out.

3. Use the principle of Pappus (inverted) to find the centre of gravity of a semicircle and of the arc of a semicircle, by reference to the volume and surface of a sphere.

4. A semicircle of 6 inches diameter revolves round an axis parallel to its diameter and 6 inches distant from it. Find the volume and surface swept out.

5. A circle of 1 inch radius moves with its centre along a helix (or screw edge) of 6-inch pitch, traced on a cylinder 3 inches in diameter, and the plane of the circle always contains the axis of the cylinder. What is the volume swept out in a whole turn?

A SELECTION OF

EXAMINATION PAPERS

ON ARITHMETIC AND MENSURATION

ACTUALLY SET AT VARIOUS PUBLIC EXAMINATIONS.

SET AT THE EXAMINATION FOR SCHOLARSHIPS AT THE ROYAL SCHOOL OF NAVAL ARCHITECTURE.

(Three hours allowed.)

Quote the number of every Question which you answer.
Show your work at full length.

1. What will be the cost of covering a space 20 yards by 12 yards with lead of 5 lbs. to the square foot at 3^1*d.* a pound?

2. Teak is 45 lbs. to the cubic foot, while water is $62\frac{1}{2}$ lbs., and the specific gravity of a certain class of iron is 7·492. What thickness of iron will be of the same weight as a 4-inch plank of teak?

3. The following is an estimate of annual cost of production of certain articles:

1,500 tons wrought iron at 16*l.* 5*s.* per ton.
75 tons wrought iron at 26*l.* 10*s.* per ton.
640 tons iron castings at 14*s.* per cwt.
45 tons gun metal at 11*d.* a pound.
Labour, 120 men at an average of 27*s.* a week. Coal, 1,200 tons at 13*s.* Superintendence, interest of capital, rent, and taxes, 4,000*l.* a year.
Contingencies 10 per cent.

What must be the selling price per ton to allow 40 per cent. profit?

4. Reduce to its lowest terms $\frac{375751}{868997}$; and find its value to five places of decimals.

5. Find the surface and cubical content of a ridge roof, length and breadth over all 60 feet and 30 feet, ridge 30 feet, and height 20 feet.

6. Ascertain the cost of 4 miles of iron piping 35 inches internal diameter, 1 inch thick. The price of iron to be reckoned at 9*l.* per ton, and its specific gravity 7·25; 5 per cent. to be allowed for overlap.

7. ABCD is a quadrilateral, of which the two diagonals AC and BD are at right angles to one another, and their lengths 160 and 116 feet respectively. Find the area in yards.

8. A conical wine-glass, 2 inches broad and $2\frac{1}{2}$ inches deep, is half-filled with water. Find the depth and quantity of water.

9. Subtract $\frac{6}{10}$ of $\frac{5}{4}$ of $\frac{7}{8}$ from $\frac{4}{13}$ of $5\frac{4}{7}$. Divide $\dfrac{2\frac{1}{3}}{9\frac{4}{7}}$ by $\frac{1}{2}$ of $\frac{6}{11}$.

Arithmetic Paper set by the Civil Service Commissioners to Candidates for Admission as Engineer Students of the Admiralty.

(Two hours and a half allowed.)

1. Write out in words the number 9000735006.
2. Write down in figures five millions fifteen thousand and five.
3. Write down in figures thirty thousand twelve hundred and eighty.
4. Add together

12489307
34567890
5729011
2357109
90463295
10639042
12839476
77690379
462183921
803672897

5. Add together

£	s.	d.
5632	4	2
4187	16	$6\frac{1}{2}$
437	15	$7\frac{1}{2}$
596	13	$1\frac{3}{4}$
5910	7	8
4728	6	11
59	12	$10\frac{3}{4}$
19403	13	8
27391	4	$6\frac{1}{2}$
40572	16	4

6. From 824019*l.* 8*s.* $4\frac{1}{2}$*d.* take 38274*l.* 13*s.* $9\frac{3}{4}$*d.*
7. Multiply 940213 by 607004.
8. Multiply 7048*l.* 11*s.* $3\frac{1}{4}$*d.* by 92.
9. Divide 560127*l.* 17*s.* $9\frac{3}{4}$*d.* by 45.
10. Divide 226125587*l.* 3*s.* $5\frac{1}{2}$*d.* by 602.
11. Reduce 2 tons 7 cwt. 2 qrs. 11 lbs. to ounces.
12. If 17 yards of cloth cost 6*l.* 12*s.* $8\frac{1}{2}$*d.*, what will $110\frac{1}{2}$ yards cost?
13. Find (by Practice) the dividend on 2870*l.* 10*s.* at 14*s.* 3*d.* in the pound.

14. Find the simple interest on 3260*l.* for 4 years at $2\frac{1}{2}$ per cent.

N.B. *The first fourteen questions should be answered before the others are attempted.*

15. What is the income corresponding to an income-tax of 59*l.* 1*s.* 3*d.* at the rate of 5*d.* in the pound?

16. In 370652308 square inches, how many acres, roods, perches, &c.?

Vulgar Fractions.

17. Add together $2\frac{1}{8}$, $5\frac{1}{2}$, $\frac{3}{20}$, and $\frac{1}{40}$.
18. Subtract $7\frac{3}{16}$ from $9\frac{1}{20}$.
19. Multiply together $3\frac{7}{8}$, $\frac{5}{34}$, $1\frac{9}{16}$, $2\frac{4}{7}$, and $1\frac{1}{10}$.
20. Divide $6\frac{1}{17}$ by $\frac{5}{21}$.

Decimals.

21. Add together ·55 of 7*s.* 6*d.* and 5·32 of 17*s.* 3*d.*, and give the answer in pence and decimal fractions of a penny.
22. Subtract 2·63 dwts. from 12·13 oz. Troy.
23. Multiply ·4302 by ·206.
24. Divide ·00256 by ·8192.
25. Divide 21·84 by ·168.
26. Express 13 minutes $7\frac{1}{2}$ seconds as the decimal of an hour.

Miscellaneous.

27. If 200 men in 12 days of 8 hours each can dig a trench 160 yards long 6 wide and 4 deep, in how many days of 10 hours will 90 men dig a trench 450 yards long 4 wide and 3 deep?

28. A person on leaving England exchanged his money for French money at the rate of 25 francs for a sovereign, and on arriving at Munich received 135 Bavarian gulden for 300 francs; what was his loss (1) in English money, (2) in French money, supposing a gulden to be worth 1*s.* $8\frac{1}{2}$*d.*?

29. Find the number of yards in the side of a square park containing 109 acres, 3 roods, 8 perches, and 9 square yards.

30. A postman has 10 miles to walk in 3 hours, and he walks the first 4 miles at the rate of $3\frac{3}{4}$ miles an hour, the next two at the rate of $3\frac{1}{3}$ miles an hour, and the next mile at the rate of 3 miles an hour; at what rate must he walk the rest in order to finish his journey at the proper time?

ADDITION (set to the same).

(*Half an hour allowed.*)

£	s.	d.	£	s.	d.	£	s.	d.
7245	6	2	8385	13	6	9035	2	10
8139	7	4	9416	9	8	4123	17	6
6904	7	3	1067	13	8	211	6	8
9065	13	8	207	4	6	94	3	4
8279	12	9	612	3	2	4321	18	9
4561	10	8	239	2	5	6409	12	8
485	2	2	2935	8	1	492	13	7
163	9	11	8019	3	8	56	4	10
752	15	3	1130	1	4	109	19	10
287	6	8	6408	13	10	9603	2	5
2307	4	2	2015	1	2	362	3	4
1528	3	6	9532	8	7	5976	2	5
691	13	9	631	7	7	1397	7	1
6127	9	10	6218	12	8	563	4	1
4201	6	8	1287	14	7	4287	9	11
232	1	11	3805	16	4	46	3	6
360	5	5	3276	5	9	210	14	8

£	s.	d.	£	s	d.	£	s	d.
134	6	6	8204	15	2	7319	2	5
2935	8	1	476	3	9	718	3	4
1316	9	8	9416	9	8	9416	9	8
3198	10	8	691	13	9	4614	3	3
1902	7	6	239	2	5	9763	2	1
9374	5	7	1067	13	8	211	6	8
752	15	3	1528	3	6	215	9	5
5736	10	7	734	5	1	6917	11	9
4321	18	9	967	2	4	7319	8	4
4917	10	8	109	19	10	9276	3	8
651	4	3	103	1	8	3276	5	9
360	5	5	8416	9	10	476	13	4
9	10	8	5618	7	6	752	15	3
9314	2	9	57	2	4	196	3	2
2376	0	9	1618	2	5	4201	6	8
346	5	5	436	2	3	976	2	5

£	s.	d.
3748	2	9
9673	1	0
163	9	11
276	8	3
1075	12	6
163	9	11
6904	7	3
46	3	6
3617	6	8
36	5	7
3198	10	8
6408	13	10
1140	15	7
1287	14	7
9171	8	4
3685	10	11

£	s.	d.
4103	0	4
1211	6	7
2244	1	1
4321	18	9
360	5	5
123	8	4
6408	13	10
2935	8	1
760	2	1
523	1	2
631	7	7
237	1	9
1414	12	4
5788	2	2
3507	6	8
56	4	10

£	s.	d.
5176	13	7
232	1	11
10	10	10
25	4	8
4201	6	8
8136	2	6
10	10	10
10	2	2
239	2	5
3257	8	11
1397	7	1
4466	1	9
563	4	1
5016	18	2
362	3	4
976	2	5

£	s.	d.
492	5	11
4016	3	9
1200	1	3
6271	18	3
9416	9	8
1067	13	8
5643	9	9
2768	12	4
5712	14	4
975	17	7
1140	15	6
3276	5	9
1397	7	1
360	5	5
10	10	10
239	2	5

£	s	d
163	9	11
203	7	1
100	0	1
4032	12	1
46	3	6
109	7	6
6409	12	8
9171	8	4
563	4	1
109	19	10
591	7	2
691	13	9
4069	13	2
301	2	1
362	3	4
1414	12	4

£	s	d.
6917	11	9
8106	2	2
2376	0	9
7682	19	4
6904	7	3
6890	12	4
4206	1	8
5188	17	3
215	9	5
2935	8	1
631	7	7
8	2	4
107	5	9
259	19	10
56	4	10
976	2	5

OXFORD LOCAL EXAMINATIONS.

JUNIOR CANDIDATES—ARITHMETIC. (*Two hours.*)

1. Add together one million five thousand and eighty, four hundred and nine thousand seven hundred and ninety, two hundred and forty-two thousand six hundred and ninety nine, and subtract nine hundred and fifty-eight thousand seven hundred and ninety-one from the sum

Explain the operation of 'borrowing' in subtraction.

2. Multiply 17943 by 5079 and divide 348753392 by 688

3. Add together 1047*l* 16*s* $4\frac{3}{4}d$, 1*l*. 0*s* $9\frac{1}{2}d$, 14*s* $8\frac{1}{4}d$, 62*l* 0*s* $1\frac{1}{4}d$, and subtract 5 tons 17 cwt. 2 qrs. 15 lbs. from 7 tons 14 cwt 2 qrs 10 lbs

4. Reduce 147*l*. 17*s* $6\frac{1}{2}d$ to farthings, add ten thousand and forty-seven farthings to the result, and express the sum in *l s. d*

5. Write out the table of avoirdupois weight, and show that 144 lbs avoirdupois = 175 lbs Troy

6. Multiply 18 tons 3 cwt 2 qrs 9 oz by 23

7. A sum of 99*l* 2*s* $9\frac{3}{4}d$ is equally divided among 47 people What does each receive? And if each subscribe one-third of his share to a fund, what will be the total amount of their subscriptions?

8. The circumference of a carriage-wheel is 7 ft 4 in How often does it turn round in a league?

9. If 15 yards of silk cost 1*l*. 13*s*. 9*d*., how much will 20 yards 1 foot cost?

10. A man buys 500 quarters of wheat at 56*s* a quarter He sells one-half of this quantity at the rate of 6*s* a bushel. At what rate must he sell the remainder so as to gain 25*l*. by the whole transaction?

OXFORD LOCAL EXAMINATIONS

SENIOR CANDIDATES.

Paper on Mensuration and Surveying, set at the end of a Trigonometry Paper. (*Three hours for both.*)

11. If the diameter of a well be 3 ft. 9 in., what is its circumference?

12 Find the volume of a regular triangular pyramid, a side of its base being 6 feet, and its altitude 60 feet.

13. Calculate the expense of making a moat round a circular island at 2*s*. 6*d*. per square yard, the diameter of the island being 525 feet, and the breadth of the moat 21 feet 6 inches

14. How many square inches of gold leaf will gild a globe 18 inches in diameter?

15. What weight of water will a rectangular cistern contain, the length being 4 feet, the breadth 2 ft. 6 in., and the depth 3 ft. 3 in., when a cubic foot of water weighs 1000 oz.?

16. Describe the cross staff and explain how by its help you would ascertain (1) the breadth of a river from one side, (2) the area of a lake.

17. Plan and find the area of a field from the following notes:

	to F	
	2718	
D	1865	953 E
B	575	1142 C
	from A	

CAMBRIDGE LOCAL EXAMINATIONS.

JUNIOR STUDENTS. (*Two hours.*)

1. Find the number which subtracted from 80,000 leaves 57735; and divide it by 365

2. Add together 7*l.* 19*s.* $6\frac{3}{4}$*d.*, 11*l.* 0*s.* 10*d.*, 28*l.* 3*s.* $4\frac{1}{2}$*d.*, and 16*l.* 8*s.* $\frac{1}{4}$*d.*, and subtract the sum from 100*l.*

3. Into how many lots can 57 tons 5 cwt 1 qr 7 lbs. be divided so that each lot contains 1 ton 12 cwt. 2 qrs 25 lbs?

4. Divide 282892 by ·394, and find the value of ·175 of 1*l.* and of 1·00375 of a ton.

5. Find the value of $\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{7}{10}$ and of $(3\frac{1}{3} + 7\frac{1}{7})$ of $\frac{2}{5} + 9\frac{3}{4}$.

6. Twenty-five years ago a man was four times as old as his son, whose present age is 33. What is the present age of the father?

7. What is the value of 5 cwt. 3 qrs. 19 lbs at 2*l.* 15*s.* per cwt.?

8. If the wages of 29 men for 54 days amount to 80*l.* 9*s.* 6*d.*, how many men must work 12 days to receive 407*l.*?

9. A person bought 374 eggs at 2 a penny and some others at 3 for twopence. He paid altogether 1*l.* 9*s.* 11*d.* How many eggs did he buy at 3 for twopence?

10. Divide 35*l.* 5*s.* $1\frac{1}{2}$*d.* among A, B, C, so that B may get twice, and C three times as much as A.

11. A mixture is made of 1 gallon of ale at 5*d.* a gallon, 3 at 9*d.*, 4 at 12*d.*, and 12 at 4*d.*; how much per gallon is this mixture worth?

12. A person invests 4700*l.* in shares which are at 98, and which pay $3\frac{1}{2}$ per cent. and the same sum in 3 per cent. Consols which are at 94. What difference would it have made to him in a year if he had invested the whole amount in shares?

13. A's capital is 200*l.* He buys goods on credit from seven persons to the amount of 40*l.* from each. He sells on credit the goods he purchased to ten persons for 330*l.* Six of his ten debtors pay him 120*l.*, and he pays 25*l.* for expenses.

Enter these transactions into the ledger, and balance the accounts.

CAMBRIDGE LOCAL EXAMINATIONS.

SENIOR STUDENTS. (*One hour and three-quarters.*)

1. Divide 69 miles 7 furl. 39 po. 2 ft. by 492.

2. Add together $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}$, and subtract the result from $100\frac{1}{100}$.

Reduce to their simplest forms

$$\frac{1}{2 + \frac{1}{3 + \frac{1}{4}}}; \quad \frac{1}{2} \text{ of } \frac{1}{3} \times \frac{3}{4} \text{ of } \frac{4}{5} \div \left(\frac{7}{8} + \frac{1}{4} \text{ of } 20\right); \quad \frac{3\frac{1}{7} - 2\frac{1}{3}}{5\frac{2}{7} + 1\frac{1}{6}}$$

3. Add together 536·421, $5364\dot{2}\cdot\dot{1}$, $5\cdot\dot{3}642\dot{1}$, and subtract the result from 100000.

$$\text{Prove that } \frac{\cdot\dot{2}\dot{1}}{\cdot\dot{2}1\dot{1}} = \frac{2331}{2321}$$

4. State your rule for the division of decimals; divide 2 by ·2, ·002 by ·02, 2·2 by 2·1.

5. Find the value of

$\frac{1}{4}$ of 17*s.* 8*d.* + 2·625 of 1*s.* − $\frac{3}{4}$ of $\frac{5}{8}$ of 5*s.* 4*d.* + $\cdot 26\dot{3}$ of 25*s.*,

and reduce the result to the decimal of 5*l.*

6. Extract the square root of 35·672 and of $\cdot\dot{4}$ to 4 places of decimals.

7. What would be the cost of paving a hall 50 yards long by 50 feet broad with marble slabs 1 foot long and 9 inches broad, the price of the slabs being 5*l.* per dozen?

8. Find the difference between the simple interest and the discount on 100*l.* for 5 years at 5 per cent.

If the present worth of 218*l.* due two years hence be 200*l.*, what is the present worth of 1000*l.* due 6 years hence at the same rate?

9. Of the boys in a school, one-third are over 15 years of age, one-third between 10 and 15. A legacy of 100*l.* can be exactly divided amongst them by giving 10*s.* to each boy over 15, 6*s.* 8*d.* to each between 10 and 15, and 3*s.* 4*d.* to each of the rest. How many boys are there in the school?

10. If the price of candles $8\frac{1}{2}$ inches long be 9*d.* per half-dozen, and that of candles of the same thickness and quality $10\frac{1}{4}$ inches long be 11*d.* per half-dozen, which kind do you advise a person to buy?

What would be the saving per cent. should your advice be followed?

THE EDUCATION DEPARTMENT.

MALE CANDIDATES FOR ADMISSION TO TRAINING SCHOOLS.

ARITHMETIC.

(*Three hours allowed for this Paper.*)

You are not permitted to answer more than *one* Question in each Section.

The *solution* must in every instance be given *at full length.* A correct answer, if unaccompanied by the solution, or if not obtained by an intelligible method, will be considered of no value.

SECTION I.

1 Add together two millions and thirty thousand, one thousand and seventy-four, thirteen hundred and nine, eighty-three thousand and five hundred, one hundred and twenty-three thousand, and take from the sum five hundred and ninety-eight thousand seven hundred and ninety-nine.

2 Divide six millions seven hundred and twenty-three thousand eight hundred and sixty-four by fifty-four thousand and eight.

SECTION II.

1. How many farthings in 59*l.* 13*s.* 6¾*d.*?

2. Taking a year as 365 days 6 hours, how many years are there in ten million minutes?

SECTION III.

1. If a pendulum vibrates 5 times in 2 seconds, how many times will it vibrate in 24 hours?

2. A servant's wages being 10*l.* 8*s.* per annum, how much ought she to receive for 7 weeks?

SECTION IV.

1. What is the rent of 23 acres 3 roods 16 poles at 2½ guineas per acre?

2. Find the expense of carpeting a room 12 ft. 4 in. long by 12 ft. 6 in. broad, with carpet ¾ yd. wide, at 4*s.* 6*d.* per yd.

SECTION V.

1. Make out this bill of parcels, and show how it should be receipted. How much change will there be out of a 5*l.* note?

$3\frac{1}{2}$ cwt. of coals at $10\frac{1}{2}$*d.* a cwt.; 13 lbs. of cheese at $7\frac{3}{4}$*d.* per lb.; $2\frac{3}{4}$ lbs. of tea at 3*s.* 4*d.* per lb.; 17 lbs. of sugar at $5\frac{1}{2}$*d.* per lb.; $3\frac{1}{2}$ yds. of flannel at 1*s.* $11\frac{1}{2}$*d.* per yd.; 29 yds. of calico at $10\frac{3}{4}$*d.* a yd.

2. A yacht and its fittings cost 1005*l.* 12*s.*; the yacht costs 5 times as much as the fittings; how much did the yacht cost?

SECTION VI.

1. A grocer buys 10 cwt. 3 qrs. 21 lbs. of sugar for 30*l.*, and pays 12*s.* 6*d.* for expenses; at what rate must he sell it per lb. to gain not less than 25 per cent.?

2. A merchant expends 1636*l.* 5*s.* on equal quantities of wheat at 2*l.* 2*s.* a quarter; barley at 1*l.* 1*s.* a quarter; and oats at 14*s.* a quarter. What quantity of each will he have?

SECTION VII.

1. Simplify $\dfrac{\frac{1}{4} + \frac{4}{5} + \frac{5}{6} + \frac{7}{15}}{\frac{1}{2} + \frac{3}{5} + \frac{7}{8} + \frac{9}{10}}$.

2. Out of a cistern two-thirds full, 20 gallons are drawn; 70 gallons are then added and it is found to be five-sixths full; how much does it hold?

SECTION VIII.

1. Multiply 5·384 by ·00723; 20·7 by 500; ·072 by 50·34; add them together and take away $\frac{1}{8}$ of ·462; what is the result?

2. Divide 500 by ·25, the quotient by ·025; and the second quotient by 50; what is the result?

SECTION IX.

1. Give the rules for multiplying and dividing vulgar fractions by vulgar fractions, and explain the reasons for the rules.

2. Give the reasons for the process of 'long division' of numbers.

3. Make notes for a lesson on simple subtraction.

MALE CANDIDATES—END OF FIRST YEAR OF TRAINING

ARITHMETIC.

(*Two hours and a half allowed for this Paper.*)

Students are not permitted to answer more than *one* question in each Section.

Acting Teachers may, but need not, confine themselves to the questions marked with an asterisk (*). They are not to answer more than *seven* questions.

The *solution* must in every instance be given at *full length*. A correct answer, if unaccompanied by the solution, or if not obtained by an intelligible method, will be considered of no value.

SECTION I.

*1. Add the following numbers: Three hundred and forty thousand and fifty, five millions nine hundred and twenty-two thousand and nine, seven hundred and four thousand three hundred and four, twenty thousand and five, sixty-five thousand six hundred; subtract from the sum three hundred and ninety-seven thousand eight hundred and ninety-seven, and write out the answer in words.

*2. How many hours are there in 10 years, of which the first is 1852?

*3. A bankrupt's assets are 613*l.* 2*s.* 8*d.*, his debts are 3698*l.* 16*s.* How much can he pay in the pound?

*4. Make out a bill of the following articles, and find the amount: 7 lbs. at 2*s.* 3*d.* per lb.; 16 lbs. at 7*d.* per lb.; 11 oz. at 1*s.* 7*d.* per oz.; 9 lbs. at 3½*d.* per lb.; 7 lbs. at 7¾*d.* per lb.; 3½ lbs. at 1*s.* 10*d.* per lb.

*5. If a silver tankard weighing 100 oz. 16 dwt. is made into spoons weighing 2 oz. 16 dwt. each, how many spoons are there?

*6. A merchant buys 10 gallons of spirits at 12*s.*, 15 at 4*s.* 6*d.*, and 18 at 5*s.* 9*d.*; what will be the price of a gallon of the mixture, so that he may gain 2*l.* 5*s.* 6*d.* on his outlay?

SECTION II.

*1. What number is the same multiple of 5 that 148995 is of 9?

2. The wages of 5 men for 6 weeks being 14*l.* 5*s.*, how many weeks will 4 men work for 19*l.*?

3. Simplify the expression $\frac{\frac{1}{2}+\frac{1}{3}+\frac{1}{4}}{\frac{2}{5}+\frac{3}{7}+\frac{9}{8}}$

SECTION III.

*1. Reduce to its simplest form $3\frac{7}{123}$ of $3\frac{4}{7} \div \frac{47}{348}$ of 9.

2. If $(\frac{2}{7} + \frac{1}{31})$ of an estate is worth 1003*l.* 17*s.* 1*d.* what is the value of $(\frac{1}{5} + \frac{1}{15})$ of it?

3. Express $\frac{3}{8}$ of 2*s.* 6*d.* + $\frac{4}{5}$ of a guinea + $\frac{5}{8}$ of 1*l.* − $\frac{1}{16}$ of a penny, as a fraction of 5*l.*

SECTION IV.

1. What decimal multiplied by 125 will give the sum of $\frac{5}{8}$, $\frac{7}{16}$, $\frac{3}{4}$, ·09375, and 2·46?

2. Reduce 14 gallons, 2 quarts to the decimal of a barrel; and 1 minute $2\frac{1}{4}$ seconds to the decimal of $\frac{1}{25}$ of a lunar month.

*3. Find the value of 45 acres 3 roods 20 perches at 111*l.* 11*s.* 4*d.* per acre.

SECTION V.

*1. Two persons gained in trade 375*l.*, one having put in 500*l.* and the other 850*l.*; what part of the profits ought each to receive?

2. If I gain 5 per cent. by selling an article for 5*l.* 5*s.*, how much shall I gain or lose per cent. by selling it for $4\frac{1}{2}$ guineas?

3. If 15 horses and 148 sheep can be kept for 9 days for 75*l.* 15*s.*, what sum will keep 10 horses and 132 sheep for 8 days, supposing 5 horses eat as much as 84 sheep?

SECTION VI.

*1. A block of stone is 4 ft. long, $2\frac{1}{2}$ ft. broad, and $1\frac{1}{4}$ ft. thick; it weighs 27 cwt.; find the weight of 100 cubic inches of the stone.

2. Extract the square root of 4020025, of ·000009, and of ·9.

3. A person rows from A to B (a distance of a mile and a half) and back again in an hour; how long would it have taken him if he had rowed equally hard, and there had been a stream of $1\frac{1}{2}$ miles an hour, running from A to B for the first 30 minutes, and then ceasing?

SECTION VII.

1. A person investing in the Four per Cents. receives $4\frac{3}{8}$ per cent. interest for his money; what is the price of the Stock?

2. A man has an income of 200*l.* a year, and an income tax is established of 7*d.* in the pound, while a duty of $1\frac{1}{2}$*d.* per lb. is taken off sugar; what must be his yearly consumption of sugar that he may just save his income tax?

8. Two persons, A and B, being on opposite sides of a wood, which is 476 yards about, begin to go round it both the same way at the same instant of time; A who is 250 yards in advance of B at starting, saunters at the rate of 11 yards per minute, and B at 34 yards in 3 minutes; how many times will A have gone round the wood before the quicker overtakes the slower?

MALE CANDIDATES—END OF SECOND YEAR OF TRAINING.

ARITHMETIC.

(*Two hours and a half allowed for this Paper.*)

Candidates are not permitted to answer more than *eight* of these questions.

The *solution* must in every instance be given *at full length.* A correct answer, if unaccompanied by the solution, or if not obtained by an intelligible method, will be considered of no value.

Algebra may be employed in solving questions in this paper. When arithmetical and algebraical solutions occur to you, you may exhibit them *both* side by side.

1. A quantity of tea is sold for 4*s.* 2*d.* per lb., the gain is 10 per cent., and the total gain is 12*l.* What is the quantity of the tea?

2. A person invested 1,911*l.* in Three per Cent. Stock at $79\frac{1}{8}$; he sold out and realised a gain of 150*l.*, after paying $\frac{1}{8}$ per cent. commission on each transaction. At what price did he sell?

3. Find to two places of decimals the value of

$$\frac{\sqrt{23}-4}{\sqrt{23}+4}+\frac{10-\sqrt{23}}{10+\sqrt{23}}$$

4. Show that

$$\frac{1}{2^2}+\frac{2}{2^3}+\frac{3}{2^4}+\frac{4}{2^5}+\frac{5}{2^6}+\frac{6}{2^7}=\cdot 9375.$$

5. What decimal fraction of $\frac{5}{7}$ of 11 qrs. 5 bush. 3 pks. is equal to $\frac{3}{8}$ of 9 qrs. 1 bush. $1\frac{1}{3}$ pk.?

Set this question in the metric system, and work it out.

6. An estate is bought at 20 years' purchase for 20,000*l.*, three quarters of the purchase money remaining on mortgage at 4 per cent.; the cost of repairs averages 180*l.* per annum. What return does the purchaser receive on his investment?

7. When the mercury in the barometer is at 30 inches, the pressure of the air on every square inch of surface is 15 lbs. What will be the

pressure on the human body when the barometer is at 31 inches (taking the surface to be 14 square feet)?

8. If railway stock bought at 28 per cent. premium pay $7\frac{1}{2}$ per cent. on the investment, how much per cent. would it pay if it were bought at 10 per cent. discount?

9. A casket contains diamonds and pearls. The diamonds are worth as many times the worth of the pearls as the pearls are worth the value of the casket. The value of the diamonds is 3,411*l.* 9*s.*, and the casket is worth 1*l.* 1*s.* What is the value of the pearls?

10. If in London I get 1*l.* for 25 francs 20 centimes, what shall I gain or lose per cent. by taking French money into Bavaria, where the exchange is 11 guld. 40 kreutzers for 1*l.*, and 8 guld. 20 kreutzers for a Napoleon? (1 guld. = 60 kreutzers.)

11. A rectangular piece of ground of 9 acres 1 rood $16\frac{1}{3}$ poles, is one-third as broad as it is long. What is (1) the distance round it? (2) from one corner to the opposite corner? Turn the question into the corresponding metric question.

12. Find to six places of decimals the sum of the series

$$\frac{2}{5} + \frac{4}{5.10} + \frac{8}{5.10.15} + \frac{16}{5.10.15.20} + \&c.$$

13. Two trains, 92 ft. and 84 ft. long respectively, move with uniform velocities on parallel rails in opposite directions; they pass each other in $1\frac{1}{2}$ second; when moving in the same directions their velocities being the same as before, the faster train passes the other in 6 seconds; find the rate at which each train moves.

EXAMINATION OF THE SOCIETY OF ARTS.

ARITHMETIC.

(*Three hours allowed.*)

1. How many casks, one of them holding $2\frac{1}{4}$ cwt., and the others each 7 cwt. 13 lbs., are required for 30 tons?

2. If 25 gross of pens are bought at 1*s.* 10*d.* a gross, and sold at 1*s.* 6*d.* a hundred, what profit is made?

3. A watch and chain are together worth 14*l.* 6*s.*, the watch being worth 8*l.* 16*s.* more than the chain; find the worth of each.

4. Bought 3 cwt. of sugar at 36*s.* 6*d.* per cwt., and reserving some for private use, sold the rest at $4\frac{1}{2}$*d.* per lb. for the amount which the whole cost; how much was reserved for private use?

5. A sets out from a certain place, and walks at the rate of $3\frac{3}{4}$ miles

an hour; B follows on horseback 20 minutes later, at the rate of $8\frac{1}{2}$ miles an hour; how soon will he overtake A?

6. What is the value of 18 gallons 3 quarts $1\frac{1}{2}$ pint at 17*s.* $10\frac{1}{2}$*d.* per gallon?

7. Divide 6*l.* 1*s.* 11*d.* into three shares, so that the second share may be four-ninths of the first, and the third three-fifths of the second.

8. If $1\frac{3}{4}$ oz. of gold, 18 carats fine, be worth 5*l.* 11*s.* 5*d.*, of what fineness must gold be in order that $6\frac{3}{4}$ dwts. of it may be worth 1*l.* 3*s.* $10\frac{1}{2}$*d.*?

9. Suppose when 13 apples weigh 4 lbs. the cost of 5 dozen apples is 7*s.* 4*d.*, what should 33 apples weigh when the cost of $4\frac{1}{2}$ dozen at the same price per lb. is 6*s.* 6*d.*?

10. Express 1*l.* 4*s.* $10\frac{3}{4}$*d.* as the decimal of 3*l.* 19*s.* 8*d.*

11 Convert 246*l.* 13*s.* 10*d.* to francs, the course of exchange being 25 francs 18 cents. per £.

12. If $8\frac{3}{4}$ lbs. of coffee, at 7 cents. 2 mils. per lb. be equal in value to $4\frac{1}{2}$ lbs. of tea, what in decimal currency is the price of the tea per lb.?

13. If the mercantile discount on a bill due in 5 months, at 4 per cent. per annum, be 1*l*, for what sum was the bill drawn?

14. Express 195*l.* 6*s* 3*d.* decimally in pounds, and then calculate on the result the difference between its simple and compound interest for 3 years at 4 per cent. per annum.

15. Which is the heavier, a pound of gold or a pound of feathers, an ounce of gold or an ounce of feathers; by how much in each case?

16. A certain number of apples were bought at four a penny, and an equal number at three a penny. The whole were sold at 7 for two-pence. Required the gain or loss per cent.

17. What change of income is made by transferring 4,275*l.* from the Four per Cents. at 80 to the Five-and-a-half per Cents. at 99?

18. If sugar can be bought at 3*l.* 14*s.* per cwt., how must it be sold to gain $12\frac{1}{2}$ per cent., after allowing a discount of $7\frac{1}{2}$ per cent.?

19. Four men can do a work in 1 hour, which two women can do in 3 hours, or six children in 2 hours. How long would 1 man 1 woman and 1 child take to do the work?

20. A piece of work can be done by A alone in 6 hours, by B alone in 5 hours, and by C alone in $4\frac{1}{2}$ hours. They all begin it together, but A only continues to work until it is finished, B leaving off 2 hours 10 minutes, and C $1\frac{1}{2}$ hour before its completion. In what time is the work accomplished?

THE METRIC SYSTEM.

(*Three hours allowed.*)

N.B.—*Only six questions to be attempted.*

Candidates may select their questions from either paper; the value of correct answers in the second paper being higher than in the first.

First or Easy Paper.

1. What is the fundamental unit in this system? Whence and why was it chosen?

2. Name the units of weight and capacity, and show how larger and smaller measures are obtained.

3. Give the English equivalents of a kilomètre, kilogramme, and kilolitre.

4. Express the value of a mile, a ton, and a gallon in kilomètres, kilogrammes, and litres.

5. How many cubic centimètres are contained in a cubic mètre?

6. A vessel measures 2 feet square and 1 foot deep, how many litres will it contain?

Second or more Difficult Paper.

7. Name those countries where the metric system is in general use, and urge any reasons for the general adoption of this system.

8. A room measures 10 mètres each way and 4 metres high; what weight of air will it contain in grammes?

9. A cistern is 2 mètres long, 5 decimètres broad, and 8 centimètres deep. What is the exact quantity and weight of water it will contain?

10. A cubical block of ice measures 3 decimètres along its edge; what will be its weight, the specific gravity of ice being ·94?

11. A cylindrical gasometer is 10 mètres in diameter, and 15 mètres high: what is its capacity expressed in litres and cbc? The volume of a cylinder $= \pi r^2 h$.

12. Explain how the units of weight, capacity, and surface have been obtained from the mètre.

MENSURATION.

(*Three hours allowed.*)

1. Find the area in yards, feet, and inches, of a room which is 17 ft. 3 in. long, and 14 ft. 7 in. wide; and the cost of flooring it at 2*s.* 6*d.* a square foot.

2. Find the area of a triangle whose sides are 28, 45, and 53.

3. Show that the area of a four-sided figure of which two sides are parallel = base × average height.

4. The sides of a quadrilateral are 3, 4, 5, and 6, 3 and 6 being parallel to one another : find its area.

5. A wall measures 94 ft. 6 in. in length, and 12 ft. 9 in. in height, and is $2\frac{1}{2}$ bricks thick : how many square rods of brickwork does it contain ?

6. What will be the expense of lining a cistern, 5 ft. 6 in. long, 3 ft. 4 in. wide, and 1 ft. 2 in. deep, with sheet lead of 12 lbs. to the square foot, the lead costing 2*l.* per cwt. ?

7. The areas of two concentric circles are 154 yards and 68 yards 4 feet : find the breadth of the annulus.

8. A semicircular protractor is 3 inches in diameter, and the rim (curved and straight) is half an inch wide : find the area of the hollow part.

9. The areas of three sides of a rectangular block of wood are 3 ft. 128 in., 18 ft. 48 n., and 6 ft 60 in.: find its volume.

10. Find the diameter and height of the least circular cylinder from which the block in the preceding question can be cut.

11. Find the surface and solid content of a regular tetrahedron whose edge is 10 feet, in feet and inches.

12. A circular vessel which is 7 inches in diameter at the top, and 6 inches at the bottom, is made to hold a pint : what must be its depth ?

13 A circular plate of an inch in thickness and a foot in diameter, is formed into a sphere : find its diameter.

14. Prove that the surface of a sphere is equal to $4\pi\,(\text{rad.})^2$

TABLES OF MONEY, WEIGHTS, AND MEASURES.

MONEY

English Money

The present unit of account is in this country the pound The coin which represents this is called a sovereign For the purposes of account the pound is divided into 20 shillings, and the shilling into 12 pence In small accounts a penny is subdivided into 4 farthings But public offices, bankers, and merchants take no account of farthings

The guinea of 21 shillings, although no longer current as a coin, is sometimes used as a nominal unit, chiefly for fees and subscriptions

The coins are the sovereign and half sovereign in gold, the crown (= five shillings), the half crown (= 2*s* 6*d*), the florin (= two shillings) the shilling, and the sixpenny, fourpenny, and threepenny pieces, in silver, the penny, halfpenny, and farthing in bronze commonly called coppers, from the material they were formerly made of There have been also issued silver penny and twopenny pieces, but these are not in ordinary circulation Half farthings and quarter farthings have also been struck for some of the colonies

The sovereign consists of 22 carat gold i e 22 parts pure gold and 2 parts copper out of 24 parts Its weight is 123 274 grains The shilling weighs 87 273 grains, as an average while new, a pound Troy being coined into 66 shillings

Tables of Money

4 farthings	=	1 penny
12 pence	=	1 shilling
20 shillings	=	1 pound
21 shillings	=	1 guinea
5 shillings	=	1 crown

Farthings			
4	Pence		
48	12	Shillings	
960	240	20	Pounds

Old Coins

Groat	=	4 pence	Mark	=	13*s* 4*d*.
Tester	=	6 pence	Carolus	=	23*s*.
Noble	=	6*s* 8*d*.	Jacobus	=	25*s*.
Angel	=	10*s*.	Moidore	=	27*s*.

Pence and Shillings.

Pence		s. d.	Pence		s. d.
20	=	1 8	140	=	11 8
30	=	2 6	150	=	12 6
40	=	3 4	160	=	13 4
50	=	4 2	170	=	14 2
60	=	5 0	180	=	15 0
70	=	5 10	190	=	15 10
80	=	6 8	200	=	16 8
90	=	7 6	210	=	17 6
100	=	8 4	220	=	18 4
110	=	9 2	230	=	19 2
120	=	10 0	240	=	one pound.
130	=	10 10			

Decimal Money.

The French unit of account is a franc, which is divided into 100 centimes. Accounts are invariably kept in francs and centimes. The sou of 5 centimes and the Napoleon of 20 francs are constantly used in common conversation and in dealing with tradesmen. The Napoleon weighs 99·564 grains Troy, and contains 89·61 grains of pure gold. The English sovereign is generally worth 25 francs 40 centimes, but for rough purposes it is reckoned at 25 francs, so that 1,000 francs represent 40 sovereigns. The Italians, Belgians, and Swiss use the French system. The United States, Mexico, and Spanish America use the dollar as their unit of account. This is divided into 100 cents or 10 dimes, but accounts are kept in dollars and cents only. The dollar is worth about 4 shillings and 2 pence, or 50 pence English, so that the cent is as nearly as possible a halfpenny. In Portugal and Brazil the nominal unit of account is the rei : the milreis or 1,000 reis is the principal unit in practice ; the Portuguese milreis is worth about 4 shillings and 8 pence, and the milreis of the paper currency of Brazil is worth about 2 shillings and 3 pence. The Russian money of account is the rouble, divided into 100 copeks. The rouble is worth 3 shillings and a penny halfpenny English.

Other Foreign Money.

British India.—The unit of account is the rupee, which is divided into 16 annas, and the anna into 12 pice. The rupee passes for 2 shillings sterling.

Prussia reckons in reichsthalers and silbergroschen. The reichsthaler is worth 2 shillings and 11 pence, and contains 30 silbergroschen

The Austrian unit of account is the gulden, containing 60 kreutzers and worth about 2 shillings.

For further information on the subject reference must be made to the class of books called Cambists or manuals of exchange. Many foreign states have an inconvertible paper currency which does not bear the same value as its nominal equivalent in metal. This paper currency fluctuates very materially The values quoted above relate to the coins, and not to their nominal equivalent in paper.

ENGLISH MEASURES OF LENGTH.

Long Measure

12 inches	=	1 foot	8 furlongs	=	1 statute mile
3 feet	=	1 yard	3 miles	-	1 league
$5\frac{1}{2}$ yards	=	1 pole	69 miles	–	1 degree (nearly)
40 poles	=	1 furlong			

Cloth Measure.

$2\frac{1}{4}$ inches	=	1 nail	4 quarters	=	1 yard
4 nails	=	1 quarter	5 quarters	=	1 ell

Inches							
12	Feet						
36	3	Yards					
198	$16\frac{1}{2}$	$5\frac{1}{2}$	Poles				
7920	660	220	40	Furlongs			
63360	5280	1760	320	8	Miles		
190080	15840	5280	960	24	3	Leagues	
4374000	364500	121500	22091	552	69	23	Degree (nearly)

The inch is further subdivided in various ways. Formerly it was divided into 3 barleycorns or 12 lines; and these divisions are still used occasionally. But it is more usual to divide the inch decimally. The league of 3 statute miles is rarely used.

Nautical Measure.—The nautical mile is reckoned as equivalent to 1 minute of latitude on the earth's surface. But this is not fixed measure, as the degrees of latitude are shorter at the equator than they are at the pole. The nautical mile may therefore vary from 6046 feet to 6104 feet, but it is generally taken as 6075 feet, which is about its length on the parallel of 45°. The nautical mile is usually called a knot. The value recently adopted by Admiralty regulation in the Navy for a knot, is a minute of longitude on the equator, which gives very nearly

1 knot = 6086.5 feet = 1855 mètres.

The fathom of 6 feet is also used as a measure of depth. There was formerly a fathom of 5 feet, but that is wholly disused. There is also the cable length of 100 fathoms, sometimes used as a nominal standard of reference.

Surveyors use a measure 22 yards long, called a chain. This is divided into 100 links, each of which is therefore 7.92 inches.

ENGLISH MEASURES OF AREA.

Square Measure.

144 square inches	=	1 square foot
9 square feet	=	1 square yard
30¼ square yards	=	1 pole, rod, or perch
16 poles		1 square chain
40 poles	=	1 rood
4 roods	=	1 statute acre
10 square chains	=	1 statute acre
640 acres	=	1 square mile

Inches							
144	Feet						
1296	9	Yards					
39204	272¼	30¼	Poles				
627264	4356	484	16	Chains			
1568160	10890	1210	40	2½	Roods		
6272640	43560	4840	160	10	4	Acres	
4014489600	27878400	3097600	102400	6400	2560	640	Mile

ENGLISH MEASURES OF CAPACITY.

Cubic Measure.

1728 cubic inches	=	1 cubic foot
27 cubic feet	=	1 cubic yard
46656 cubic inches	=	1 cubic yard
35 cubic feet	=	1 ton of sea water.

Corn Measure.

2 pints	1 quart	8 gallons	=	1 bushel
2 quarts	1 quartern	5 bushels	=	1 sack
2 quarterns	1 gallon	10 bushels	=	1 quarter
2 gallons	1 peck			

Corn is generally quoted by measure, but is actually sold by weight, the gallon being reckoned as 7 lbs., the bushel as 56 lbs., and the quarter as 5 cwt. Wheat usually weighs from 60 to 64 lbs. to the bushel measure, the best being the heaviest. Various local measures are still used in many corn markets. The old corn-gallon contained 268·8 cubic inches.

Coal Measure (disused).

3 bushels	=	1 sack
12 sacks	=	1 chaldron

Coal is now sold by weight; but coke is still sold by the old coal measure.

Liquid Measure.

4 gills, quarterns, or noggins	=	1 pint	
2 pints	=	1 quart	
4 quarts	=	1 gallon	
9 gallons	=	1 firkin	beer
18 gallons	=	1 kilderkin	beer
36 gallons	=	1 barrel	beer
54 gallons	=	1 hogshead	beer
42 gallons	=	1 tierce	spirits
63 gallons	=	1 hogshead	spirits
2 hogsheads	=	1 pipe	
2 pipes	=	1 tun	

A hogshead of wine varies from 48 to 60 gallons. A pipe of wine varies from about 96 to 120 gallons. Sherry is also sold by the quarter cask of 27 gallons nearly. Brandy and bottled ale are also sold by the

'*reputed pint*' or '*reputed quart*' of 12 and 6 to the gallon respectively. Wine and brandy are also sold by the *dozen*, which is taken to be 2 gallons. The Imperial gallon contains 277·27 or 277·274 cubic inches. The old wine gallon (or Winchester measure) contained 231 cubic inches, and the old ale gallon 282. A pipe is also sometimes called a butt. The butt of ale is 108 gallons.

ENGLISH MEASURES OF WEIGHT.

Avoirdupois Weight.

16 drams	=	1 ounce (oz.)
16 ounces	=	1 pound (lb.)
14 pounds	=	1 stone
2 stones	=	1 quarter (qr.)
4 quarters	=	1 hundredweight (cwt.)
20 hundredweight	=	1 ton

Ounces					
16	Pounds				
224	14	Stones			
448	28	2	Quarters		
1792	112	8	4	Cwt.	
35840	2240	160	80	20	Ton

Troy Weight.

Used for gold and silver, jewellery, and some chemical experiments.

24 grains	=	1 pennyweight (dwt.)
20 pennyweights	=	1 ounce (oz.)
12 ounces	=	1 pound (lb.)

Apothecaries Weight.

Used for medical prescriptions only.

20 grains	=	1 scruple
3 scruples	=	1 dram
8 drams	=	1 ounce
12 ounces	=	1 pound

These denominations are not all used together. Thus heavy weights

are reckoned in tons, cwt., quarters, and lbs. Stones, pounds, and ounces are used for weights usually ranging below 2 cwt. Butchers use a stone of 8 lbs.

The pound, ounce, and grain are the same in Troy as in apothecaries weight. Practically, pennyweights are not now used. Bullion is purchased at the Mint in ounces Troy and decimal parts of an ounce.

An ounce Troy contains 480 grains, and a pound Troy 5760 grains. An avoirdupois pound contains 7000 Troy grains.

ENGLISH UNITS OF WEIGHT AND MEASURE.

The Units of weight and measure depend upon a standard yard and a standard pound Troy, which were made by order of Parliament in 1760. All other English weights and measures depend upon these. But in case of loss or injury to the standard, it was declared that the length of the pendulum vibrating seconds in vacuo in London at the sea level should be taken to be 39·13929 inches, so that it should always be possible to find the length of the inch by proper experiment and calculation. In the same way the weight of a grain may be determined from the fact, that a cubic inch of distilled water weighed in air at a temperature of 62 degrees Fahrenheit and with the barometer at 30 inches, is declared to be 252·458 grains. The pound avoirdupois contains 7000 grains, and the pound Troy 5760 grains. The Imperial gallon has been doubly defined as being 10 pounds avoirdupois of distilled water under the conditions mentioned above, or 277·274 cubic inches.[1] This contains by implication the whole comparison between English standards of weight and measure. The exact weight of a cubic foot of fresh water under these conditions is 62·3535 lbs. or 997·656 oz. avoirdupois. Now ordinary fresh water is usually a little heavier than this, from its containing air and mineral matter in solution. It is therefore convenient to reckon a cubic foot of ordinary fresh water as 6¼ gallons, 62½ lbs. or 1000 oz. : 16 cubic feet are thus 100 gallons. The exact number of cubic feet in a ton of water is 35·943, and may therefore be taken as 36 cubic feet. In ship-building a ton of sea water is reckoned at 35 cubic feet, or 64 lbs. to the foot.

It is worth while to mention that the ton of shipping is not a fixed quantity. A ton of cargo simply means the quantity which the shipowner is willing to take *as a ton* for the purpose of charging freight, and this is different for nearly every article of trade. A ton of *tonnage* merely means a certain number of cubic feet which a vessel is supposed

[1] The definition of a gallon as 277·274 cubic inches has since been repealed, and Professor Rankine affirms that 277·271 is more exact.

to have disposable for stowing cargo, on a very artificial measurement, which differs in different countries, and for different purposes in the same country. The only fixed thing is that a ton weight of salt water measures 35 cubic feet very nearly, and is taken to have exactly this measure.

MEASURES OF TIME.

The civil day is measured from midnight to midnight. This is in reality a variable period. An average day, of mean length, is therefore taken as the fixed period which represents the civil day.

The solar year is the time of a whole revolution of the earth in its orbit. It is 365·24222 days. The civil year is generally 365 days; but every fourth year is made 366 days, which brings the civil year into a rough agreement once in four years. The leap year is the year of which the number is divisible by 4. Thus 1764 and 1868 are leap years, but not those which leave a remainder to 4. To complete the adjustment, three leap years have to be dropped in four centuries, and this is done by omitting the 'leap-day,' which is always the 29th of February, from the year of the century unless the number of the century is also divisible by 4. Thus the years 1600, 2000, and 2400 are leap years, but not 1700, 1800, 1900, or 2100. This is according to the Calendar settled by Pope Gregory XIII. in 1582, and introduced into England by Act of Parliament in 1752. Previously to that the Julian Calendar had been in use, under which all years of which the number is divisible by 4 (including the centuries) were leap years. In passing from the *old style* to the *new style*, 10 days were dropped by Pope Gregory, and 11 days by the English, to bring the Calendar in accordance with the date of the vernal equinox in A.D. 325. The difference between the old and new styles is now 12 days.

The Russians and Greeks still reckon by the old style. Their reckoning consequently lags 12 days behind ours. Thus our 22nd January is their 10th January.

The year is divided somewhat arbitrarily into 12 months of different lengths, which may be remembered by the following rhyme:

Thirty days have November,
April, June, and September,
The other month have thirty-one,
February has twenty-eight alone,
But leap year coming once in four,
Gives to February one day more.

A lunar month, that is the mean time which elapses between two successive new moons, or two successive full moons, is 29·5306 days.

The month for arithmetical purposes has been explained in p. 38. The week is always seven days.

The day is divided into 24 hours; the hour into 60 minutes; and these again into 60 seconds. When further divisions are needed, decimals are used. The table is as follows:

60 seconds	=	1 minute
60 minutes	=	1 hour
24 hours	=	1 day
7 days	=	1 week
*30 days	=	1 month
12 months	=	1 year
*365 days	=	1 year

Those marked * are inaccurate, as already stated, and the 12 months to a year is only true of the 12 calendar months of variable length

The *civil* day is reckoned from midnight to midnight.

The *nautical* day is from midday to midday *preceding* the civil day.

The *astronomical* day is from midday to midday *following* the civil day.

Thus 2 A.M. on August 22, by civil time, is the 14th hour of August 22 by nautical time, and the 14th hour of August 21 by astronomical time.

The *time* at any place is mean solar time of the place itself. That is to say, it is legally twelve o'clock when the sun is due south (or north in southern latitudes) at the place, correction being made for the irregular length of the day. But throughout Great Britain it is now usual to reckon by Greenwich time, although that is not yet legalised. In France, Paris time is used.

Ships go by their own solar time, and consequently lose a day in their reckoning if they go round the earth westward, and gain a day eastward. There is thus a day's difference in the reckoning on the two sides of some (unsettled) meridian in the Pacific Ocean among English spoken people, as the Sandwich Islanders take their reckoning from England *viâ* America, and the New Zealanders from England *viâ* the Cape of Good Hope.

The date of the year is reckoned from the supposed date of the birth of Jesus Christ. I say the supposed date, because there appears to be an uncertainty to the extent of three or four years on this point, and in consequence of this there is also an uncertainty of three or four years in the comparison of Christian dates with those of Pagan Rome, which were reckoned from the supposed date of the building of the city. This date is generally taken to be 752 years before Christ.

The Russian and Greek reckoning, as already remarked, differs by 12 days from our own.

The Jews have a calendar of their own. They reckon from what they suppose to be the date of the creation of the world, namely, 3760 before Christ.

The Mahommedans have also a separate calendar. Their reckoning dates from the Hegira, or flight of Mahommed from Mecca, A.D. 622.

The year 5632 of the Jewish era commenced on the 16th Sept. 1871, and the year 1288 of the Mahommedan era commenced on the 23rd March, 1871.

Towards the close of the last century the French Republicans endeavoured to remodel the calendar after a manner which has completely fallen into disuse. But it is as well to mention that the year One of the Republic began on the 22nd September, 1792. This calendar was discontinued in France on the 31st December, 1805.

DIVISION OF THE CIRCLE.

The most important division of the circle is into 4 equal parts, called quadrants; the next in geometrical importance is its division into 6 equal arcs, the chord of each of which is equal to the radius. But it was the practice of the Egyptian astronomers to divide the whole circle into 360 parts called degrees; each of these degrees was divided into 60 parts called minutes; these again into 60 equal parts called seconds; and so on. To this division we still adhere, except that it is usual to subdivide the seconds decimally, and not to use their 60th parts called *thirds*. The table is as follows:

60 seconds make 1 minute
60 minutes make 1 degree
90 degrees make the quadrant
360 degrees make the complete circle.

Seconds				
60	Minutes			
3600	60	Degrees		
324000	5400	90	Quadrants	
1296000	21600	360	4	Circle

THE METRIC SYSTEM.

Shortly before the French Revolution of 1789, the confusion of weights, measures, and coins current in France was beyond anything that at present exists in any part of Europe. It was quite by accident if two neighbouring towns used either the same standard or the same mode of reckoning, and there was no one standard of sufficiently general use to be reasonably adopted at the sacrifice of the others. Under these circumstances, the French Legislative Assembly took the resolution of establishing an entirely new system of measurement, coherent in all its parts, and depending for its primary unit on a natural line, liable to no alteration so long as the earth remained the same. They further resolved that the whole of the divisions and multiples of all their units should be decimal.

The natural standard which they adopted as the base of their primary unit was the quadrant of latitude; that is, to say, the distance of the pole of the earth to the equator measured at the level of the sea. The actual unit of measure was taken as one ten-millionth of this quadrant, and it was called a mètre. Every other unit of the metrical system depends upon this. The multiples of the units were to be denoted by the Greek numerals, and their divisions by the Latin numerals, according to the following scale:

Multiple	Name	Derivation	Meaning
		Greek	
10000	myria-	μυριάς	ten thousand
1000	kilo-	χίλιον	a thousand
100	hecto-	ἑκατόν	a hundred
10	déca-	δέκα	ten
Divisions		*Latin*	
$\frac{1}{10}$	déci-	decem	ten
$\frac{1}{100}$	centi-	centum	a hundred
$\frac{1}{1000}$	milli-	mille	a thousand

This scale ranges over ten millions of times the lowest denomination. The choice of the actual unit in each case to which this scale is to be applied was settled by motives of supposed convenience. The unit of length, as already stated, was the mètre, between 39 and 40 English inches in length. The unit of land measure is the *are*, a square of 10 mètres each side. The unit of cubic measure is the *stère*, or cubic mètre; for liquids the *litre*, which is a cubic décimètre. For weight,

the *gramme*, which is the weight of a cubic centimètre of distilled water at freezing point.

Only a certain number of these 50 weights and measures are in common use, for the metrical system has not been absolutely adopted even in France. It is much more common to hear of a mètre cube than of a stère, and hectomètres and hectogrammes are seldom if ever mentioned. The choice of standards was not always happy, especially in the case of the gramme, which was too small; and even their primary standard (which is a certain bar of platinum deposited in the archives of France) has recently been stated on high authority not to be quite accurately what it is supposed to represent, viz. a ten-millionth part of a quadrant of meridian.

The system of decimal division was also applied to the circle; the quadrant was divided into 100 grades, the grades into 100 minutes, and the minute into 100 seconds. But this has been long since entirely abandoned, except by surveyors, although suitable tables have been calculated. It does not appear that any serious attempt was made to divide time decimally, although the calendar was remodelled.

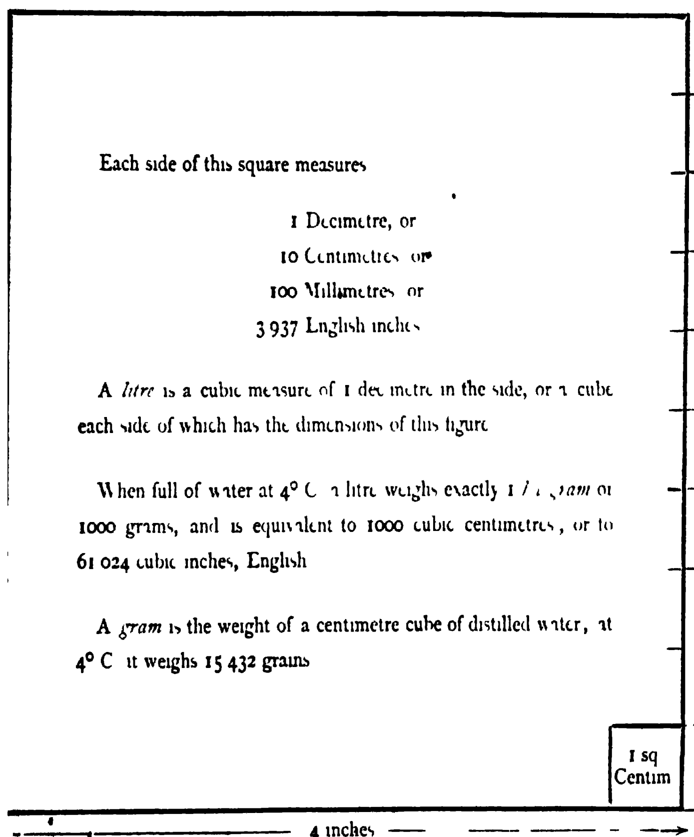
The following table contains the weights and measures chiefly used, with their English equivalents:

1 yard	=	·91438347 mètre
1 foot	=	·30479449 mètre
1 inch	=	·02539954 mètre
1 mile	=	1·6093149072 kilomètre
1 grain Troy	=	·06479895 gramme
1 pound avoir.	=	·4535926525 kilogramme
1 ton	=	1·0160475416 tonne
1 acre	=	·4046711 hectare
1 gallon	=	4·54102 litres
1 mètre	=	1·09363307 yard
1 mètre	=	3·2808992 feet
1 mètre	=	39·3707904 inches
1 kilomètre	=	·621383 mile
1 gramme	=	15·43234874 grains
1 kilogramme	=	2·20462125 lbs. avoir.
1 hectare	=	2·47114 acres
1 litre	=	·220215 gallon

The tonne is used for 1000 kilogrammes, and does not differ very much from the English ton. It is the weight of a cubic mètre of water.

A rough rule for converting French mètres into English yards is to add 10 per cent to them Thus 40 metres are very nearly 44 yards

It may facilitate, to some minds, the realisation of what French measures actually are, to reproduce the accompanying diagram, taken from Miller's 'Inorganic Chemistry,' a work of this series



The entire square is the 100th part of a square metre, and contains 15 5 square inches, or 1076 of a square foot nearly

MEASUREMENTS DEPENDING ON THE EARTH.

The earth is approximately a sphere, but only approximately so. It is nearer the truth to describe it as an oblate spheroid, that is, a figure traced out by an ellipse revolving round its shortest axis. The best values for its dimensions appear to be those given by Captain Clarke :

	Feet		Mètres
Equatorial diameter	41847662	=	12754937
Polar axis	41707536	=	12712227

A recent writer remarks that the equator is not a circle, but an ellipse, of which the axes differ by about 10000 feet. The mean value is given above. When we come to a variation of 2500 feet from a mean value of the radius, we get into difficulties about seas and mountain tracts.

The mean density of the earth is considered to be about 5 or 5½ times that of water. The mean density of rocks at the surface is about 2½ times that of water.

The attraction of the earth at its mean surface varies from the pole to the equator. Its effect on falling bodies is reckoned as giving a velocity in one second of 32·187 feet per second (in London), or 9·8087 mètres per second (in Paris). It is commonly taken as 32·2 feet English.

The pressure of the atmosphere, with the barometer standing at 29·92 inches of mercury or 33·9 feet of water, is 14·7 lbs. on the square inch

SPECIFIC GRAVITIES AND WEIGHTS PER CUBIC FOOT.

(Water being taken as unity, except for gases.)

Gases at 32° Fahrenheit and Atmospheric Pressure.

	Sp. gr.	Weight
Air	1	·0807
Carbonic acid	1·53	·1234
Hydrogen	·07	·0056
Oxygen	1·16	·0893
Nitrogen	·97	·0786
Steam at 212°	·47	·0379

Liquids.

	Sp. gr.	Weight
Pure water	1	62·425
Sea water	1·025	64
Alcohol, proof	·916	57·2
,, pure	·791	49·4
Sulphuric acid (ordinary)	1·8	116
Mercury	13·6	848·75

Metals

Iron, cast	7·11	444
Iron, wrought	7·7	480
Steel	7·9	490
Copper	8·8	550
Lead	11·4	712
Tin	7·4	460
Zinc	7	437
Silver	10·5	655
Gold	19	1190
Platinum	21·5	1344

Various Materials for Building.

Glass and compact rocks, from	2·5	156
Glass and compact rocks, to	3	187
Masonry (average)	2	125
Brickwork, dry	1·8	112
Sand, dry	1·4	88
Sand, wet	1·9	118

Timber, dry.

Willow	·4	25
Elm	·54	34
Teak, Indian	·7	45
,, African	1	62·5
Oak (average)	·8	50
Fir, from	·48	30
Fir, to	·7	44
Cork (bark)	·24	15

Other Substances.

	Sp. gr.	Weight
Common salt . .	2·13	133
Sulphur . .	2	125
Coal (average) .	1·25	78
Gunpowder (solid) .	2	125
,, (loose) .	·9	56

Most of these things vary within very wide limits, and many of those for which the dry weight is given usually contain moisture.

ADDITIONAL NOTES.

It is worth while to explain certain numerical terms which occur with more or less frequency in English as well as foreign arithmetic.

DOZEN.—This means 12; but it is customary in certain trades to sell 13 or 14 to the dozen. A 'baker's dozen' is generally understood to be 13, although in reality it is more often 14.

SCORE.—20.

GROSS —12 dozen, or 144. There is also what is called a *great gross*, of 72 dozen, although this term is now seldom used.

HUNDRED.—Strictly 100 but the old-fashioned hundred, still used in selling fish, and sometimes fruit, is six score, or 120.

MYRIAD.—A Greek term for 10,000.

LAC.—An Indian term for 100,000.

CRORE.—An Indian term for a million (1,000,000).

Engineers and artificers frequently use, for shortness, a single dash to express feet, and two dashes to express inches. Thus 2′ 3″ means 2 feet 3 inches. This notation is almost always used on plans and working drawings.

ANSWERS TO EXAMPLES.

PAGE

5. 1. 30,019,004.
2. 103,470,860.
3. 3,000,000,000,009.
4. 990,909.
5. Eighty millions seven hundred and nine thousand and fifty-three.
One hundred millions and ten.
Ninety millions nine hundred and nine thousand and ninety.
Nine millions, ninety thousand, nine hundred and nine.
One hundred and seventy thousand.
Seven millions, seven hundred and seventy-seven thousand and seventy-seven.

8. 6. 94, 116, 1767.
19, 54, 1349.
7. XLV, XVIII, CCCCXCIX, XLIX, MDCCCLXX, DLV.

12. (1) 3,021.
(2) 1,982.
(3) 14,237,088.
(4) 4,148,484.
(5) 1,111,111,110.
(6) 1,000,000.
(7) 91,101,110.
(8) 20,827,016.
(9) 12,002,222.
(10) 4,702,770.

(11)

3,509,034
40,490,403
37,000,430
30,406
700,008
895
22,728,863
104,460,039

PAGE

13. (12)

875,035
5,902
301,934
1,974,610
39,000
5,742
8,799,999
12,002,222

(13) Cols.

Cols.		lines	
A	22688	*a*	7207
B	21841	*b*	19332
C	19903	*c*	22494
D	19640	*d*	24042
E	35416	*e*	22537
F	29571	*f*	8275
	149059	*g*	19394
		h	25778
			149059

15. (1) 52. (2) 121. (3) 17. (4) 34.

16. (5) 890819. (6) 5228787.
(7) 864197532. (8) 434372550488818.
(9) 999999999. (10) 98888889.
(11) 1000990909. (12) 737080720352222.

18. (1) 2173. (2) 60890. (3) 2796203.

24. (12) 1436226624. (13) 2872556494008787.
(14) 1944460921158. (15) 5060344127169150.
(16) 29622961181458719148o656.
(17) 1020302907866668809303020l.
(18) 161853049910276611 2.
(19) 9999999500000040000.
(20) 999400149980001499940001.

25. (21) 999400149980001499940001.
(22) 231878583553 6.
(23) 45673337928960.
(24) 121932631112635269.

30. (10) 8009909.
(11) 111111111, 10101010, 1001001, 100010,
90909090, 9900990, 999000, 99990,
76923076, 27027027, 16393442, 2652519,
142857142, 10989011, 7462686, 1953125,
40000000, 1600000. 512.

PAGE

31. (12) 48481368. (13) 3183098862.
(14) 4342944819. (15) 2302585093

39. 1. 65998. 2. 180 and 111 farthings over
3. 5223360. 4. 27 tons 18 cwt. 0 qrs. 4 lbs
5. Taking the present date as October 1, 1871, we have 1870 years (453 of which are leap years) and 273 days. The total number of days is thus 683276 days=97610 weeks and 6 days.
6. Taking the years as 365 days, it is 31536000 secs.
7. 0 years 115 days 17 hrs. 46 min. 40 sec.
8. 907872 grains. 9. 454380.
10. 173 lbs. 3520 grains. 11. 32345½.
12. 18 m. 7 fur. 20 p. 10 ft. 13. 648000.
14. 27878400. 15. 46656.
16. 578 ft. 1216 in. 17. 441 guineas 8*s.*
18. 8400.

40. 19. 288. 20. 6260933¼.
21. 3 a. 0 r. 5 p. 23¾ sq. yds. 22. 17280.
23. An avoirdupois ounce is 437½ grains troy. The cubic foot of water is therefore 437500 grains. Dividing by 1728, the cubic inch of water weighs $253\frac{316}{1728}$ grains, or nearly 253 grains troy.
24. 108 gallons=1728 half-pints.
25. 35 cubic feet and 1568 cubic inches.

42. (1) 779*l.* 1*s.* 4*d.* (2) 870 tons 1 cwt. 3 qrs. 5 lbs. 6 oz.
(3) 70 lbs. 1 oz. 10 dwts. 11 grs. (4) 966 a. 1 r. 20 p.

43. (5) 66 miles 1639 yards. (6) 747 yards 2 qrs. 6 in.
(7) 15 ft. 2 in. (8) 387 sq. ft. 1 sq. in.
(9) 1337 dollars 15 cents.
(10) 1871 francs 35 centimes.
(11) 1062 tons 19 cwt. 1 qr. 26 lbs. 8 oz.
(12) 66 miles 9 p. 4 ft. 10 in.=348633 ft. 4 in.
(13) 29 a. 137 yds. 8 ft. 64 in. (no odd chains).
(14) 19 yards 1360 in.=887824 inches.

PAGE

45. 1. 2*l.* 14*s.* 10*d.* 2. 807 yards.

46. 3. September 30, 1831 (see next question).

4. He was 260 days old on June 17, 1832, which is a leap year, and that date is therefore the 169th day of the year. He was born on the 274th day of 1831, or on October 1, 1831.

5. 6 cwt. 1 qr. 16 lbs. 6. 6 oz. 5 dwts. 20 grs.

7. 6*l.* 8*s.* 4*d.* richer: least capital 1*l.* 13*s.* 4*d.*

9. 10 miles 1560 yards east.

48. 3. 150 tons 15 cwt. 0 qr. 14 lbs.

49. 4. 272*l.* 12*s.* 11*d.* 5. 57 tons 5 cwt. 1 qr. 21 lbs.

6. 253 tons 9 cwt. 3 qrs. 16 lbs. 7 oz.

7. 113 lbs. 9 oz. 6 dwts.

8. 330 m. 3 fur. 25 yards. 9. 11017 a. 3 r. 37 p.

10. 3054 sq. ft. 27 sq. in.

11. 261 yards 2 qrs. 2 nails.

12. 11 cubic yards 1 ft. 956 in.

13. 30 miles 7 furlongs.

14. 382 miles 7 fur. 91 ft. $10\frac{1}{2}$ in.

15. 375 ft. per hour. 16. 18662400.

17. 38 miles 320 yards.

18. 8 tons 17 cwt. 1 qr. 23 lbs.

19. Wt. 7000 tons 14 cwt. 1 qr. 4 lbs.; cost 522720*l.*

20. Wt. 1841 tons 10 cwt. 1 qr. 12 lbs.; cost 1383*l.* 6*s.* 8*d.*

51. 1. 28571*l.* 8*s.* $6\frac{3}{4}$*d.* 2. 1369*l.* 17*s.* 3*d.*

3. 2 lbs. $15\frac{23600}{30000}$ oz. or 2 lbs. $15\frac{59}{75}$ oz.

52. 4. 1 cwt. 1 qr. 1 lb. = 141 lbs.

5. $6\frac{4160}{5280}$ oz. = $6\frac{26}{33}$ oz.

6. 1 a. 2 r. 35 p. $27\frac{1}{2}$ sq. yards. 7. 25*s.*

8. 10 widths 25 yds. long = 250 yds. run: cost 68*l.* 15*s.*

9. 6680.

10. 78 to a ton: 13587 in 173 tons 5 cwt.

11. 8 miles 2660 feet.

PAGE

52. 12. 768. Note that there are four rails to a double line

13. $5\frac{5}{8}$ and a trifle more. 14. $14\frac{34}{35}$, or nearly 15 feet.

15. 2 feet $4\frac{1}{2}$ in. nearly. 16. 3437 a. 0 r. 20 p.

17. 220 ft. 9 in. We assume that the poles are equidistant, and that there is not a pole at each end.

18. 12 lbs. 7 oz.

53. 19. 3010 ft. 4 in. 20. 12000.

62. (1) $\frac{22}{7}$. (2) $\frac{[illegible]}{9}$. (3) 31.
(4) $\frac{65}{9}$. (5) $\frac{871}{48}$. (6) $\frac{4960}{71}$.
(7) $\frac{724249}{729}$. (8) $\frac{1023415}{1001}$. (9) $\frac{111710}{1141}$.
(10) $\frac{10092}{100}$. (11) $\frac{10028}{101}$. (12) $\frac{1003000}{1001}$.
(13) $\frac{1111111112}{9}$. (14) $\frac{1013}{13}$. (15) $\frac{1011}{11}$.
(16) $\frac{1007}{7}$. (17) $\frac{10000000}{9}$. (18) $\frac{1000000}{99}$.

2.

(1) $143\frac{1}{7}$. (2) $4\frac{1}{17}$. (3) $3\frac{11}{15}$.
(4) $6\frac{17}{69}$. (5) 13. (6) $2\frac{569}{2735}$.
(7) 7. (8) $2\frac{242}{821}$. (9) $125\frac{704}{7057}$.
(10) $3\frac{429602}{743027}$. (11) $2\frac{10411581}{10411385}$. (12) $2\frac{10000053}{41263491}$.
(13) $60\frac{1438371}{1438471}$. (14) 1601 (15) $3199\frac{106}{1483}$
(16) $2\frac{2847761}{7843263}$.

3.

(1) $\frac{10}{5}$ and $\frac{62}{31}$. (2) $\frac{77}{11}$. $\frac{91}{13}$ and $\frac{1001}{143}$.
(3) $\frac{1573}{11}$. (4) $\frac{99}{11}$.
(5) $\frac{9999}{11}$ and $\frac{91809}{101}$. (6) $\frac{999}{9}$ and $\frac{9999}{99}$.

4.

(1) $\frac{18}{4} = 4\frac{1}{2}$. (2) $\frac{25}{8} = 3\frac{1}{8}$. (3) 16.
(4) 169. (5) 2. (6) 5005.
(7) 2890289.

PAGE

5.

63. (1) $\frac{3}{24} = \frac{1}{8}$. (2) $\frac{1}{8}$. (3) $\frac{8}{50} = \frac{4}{25}$.

(4) $\frac{1}{5929}$. (5) $\frac{107}{15039946}$. (6) $\frac{1}{2}$.

(7) $\frac{1}{169}$. (8) 1. (9) $\frac{289}{1001}$.

(10) $\frac{3682778}{1001 \times 7 \times 731} = \frac{5038}{7007} = \frac{458}{637}$.

6.

(1) $\frac{3}{2}$. (2) 18. (3) 15. (4) 5329.

(5) 18769 (6) 99990. (7) 999900.

(1) 20*l.* 15*s.* 6*d.* (2) 22*l.* 10*s.*

(3) 29*l.* 0*s.* 6*d.* (4) 15 cwt.

(5) 1 ton 6 cwt. 1 qr. (6) 7 cwt. 2 qrs.

(7) 4 cwt. 0 qrs. 11 lbs. 6 oz. (8) 400 yards.

(9) 248 acres 5 chains 460 sq. yards.

(10) 8263366 miles 66 acres and a fraction.

(11) $82\frac{1}{42}$ grains

(12) 1 ton 14 cwt. 0 qrs. 23 lbs. 3 oz.

8.

(1) $\frac{15}{47}$. (2) $\frac{11}{24}$. (3) $\frac{3}{4}$. (4) $\frac{169}{693}$.

(5) $\frac{22}{13}$. (6) 99. (7) $\frac{109791}{86359}$.

(8) $\frac{637}{323}$. (9) $\frac{53}{29}$. (10) $19\frac{371}{703}$.

9.

(1) 14. (2) $\frac{10}{7}$. (3) $\frac{4}{3}$.

(4) 169. (5) 26645. (6) $3753\frac{4}{5}$.

(7) $37\frac{8}{9}$. (8) $\frac{1}{341}$. (9) $9\frac{1635}{1813}$.

10.

(10) $\frac{1001}{10001}$.

11

64 (1) $\frac{8}{11}$. (2) $\frac{13}{28}$. (3) $\frac{29}{62}$. (4) $\frac{58}{85}$.

PAGE

12.

64. (1) $\frac{3}{4}, \frac{28}{37}, \frac{25}{33}, \frac{22}{29}, \frac{19}{25}, \frac{16}{21}, \frac{13}{17}, \frac{10}{13}, \frac{7}{9}, \frac{4}{5}$.

(2) $\frac{8}{25}, \frac{25}{78}, \frac{17}{53}, \frac{26}{81}, \frac{9}{28}, \frac{28}{87}$.

14.

(1) $\frac{19}{12}$. (2) $\frac{103}{60}$. (3) $\frac{659}{323}$.

(4) $\frac{53}{70}$. (5) $\frac{83}{100}$. (6) $\frac{13}{12}$.

(7) $\frac{5}{12}$. (8) $\frac{311}{1001}$. (9) $\frac{129}{1001}$.

(10) $\frac{157}{1001}$. (11) $\frac{311}{1001}$. (12) $\frac{64}{10001}$.

(13) $\frac{369}{700}$. (14) $\frac{169}{323}$. (15) $\frac{7381}{2520}$. (16) $\frac{1627}{2520}$.

75. 1. $\frac{219}{363} \times 14l.\ 12s. = 8$~~[illegible]d.~~ 8£ — [illegible]

2. $\frac{92}{7} \times 18s. = 11l.\ 16s.\ 7d.$

3. $\frac{100l \times 7}{18s.} = \frac{7000}{9} = 777\frac{7}{9}$ days.

4. $\frac{38 \times 336}{8}$ pence $= 6l.\ 13s.$

5. $\frac{95}{11} \times 3s.\ 6d. = 1l.\ 10s.\ 2\frac{3}{4}d.$

6. $\frac{1\frac{1}{4}}{15} \times 27$ tons $= 2$ tons 5 cwt.

7. $\frac{8}{9} \times 190$ lbs. $= 168\frac{8}{9}$ lbs.

8. $\frac{13}{15} \times 65$ ft. $= 56$ ft. 4 in.

9. $\frac{18}{25} \times 35s. = 25s.\ 2\frac{1}{2}d.$

10. $\frac{9680}{1000} \times 162$ pence $= 6l.\ 10s.\ 8d.$

11. $9 \times 9 \div \frac{3}{4} = 108$ yards.

12. The gain on each herring is, in pence,

$$\frac{54}{100} - \frac{32}{120} = \frac{82}{300}.$$

Thus $120 \div \frac{82}{300} = 439$ the answer.

13. $\frac{1728}{195} \times 59$ lbs. $= 522$ lbs. 13 oz. The $62\frac{1}{2}$ lbs. has nothing to do with the sum.

76. 14. $\frac{141 \times 56}{22\frac{1}{2}} = 350\frac{14}{15}$.

15.

	Cwt.	qrs.	lbs.	oz.
Lead	1	1	22	12
Tin	2	1	19	4
Bismuth . . .	3	3	14	0
Alloy	7	3	0	0

PAGE

76. 16. $1214\frac{4}{5}$. 17. $366\frac{1}{4}$.

18. $\frac{5}{4} \times \frac{5}{4} \times 20 \times \frac{4}{3} \times \frac{4}{3} = \frac{500}{9} = 55\frac{5}{9}$ feet.

19. $\frac{24 \times 9 \times 2}{42 \times 18 \times 12} = \frac{1}{21}$ of an inch.

20. $\frac{120 \times 33000 \times 12 \times 60}{140 \times 6 \times 10} = 678857\frac{1}{7}$ gallons.

21. $\frac{21}{11\frac{4}{10}} \times 112 = 206\frac{6}{19}$ lbs.

22. $\frac{21}{11\frac{4}{10}} \times 1200 = 2210\frac{10}{19}$ cub. in.

23. $\frac{49}{100} \times \frac{33}{47} \times 54 = 18\frac{2718}{4700}$ gallons.

24. $\frac{15000 \times \frac{365}{2}}{1825 \times 60} \times 73$ tons, &c. $= 25 \times 73$ tons, &c.
$= 1837$ tons 1 cwt. 0 qrs. 8 lbs.

25. $32 \times 3\frac{4}{7} = 114\frac{2}{7}$ feet per second.

26. This is not a sum in proportion, because the body does not fall through the same space in each second. On this subject see Chapter XVI. The true answer is 10 seconds nearly.

77. 27. The coach has $\frac{5}{4} \times 8\frac{1}{2}$ miles' start; the horseman overtakes this at the rate of $5\frac{1}{2}$ miles an hour. The time is therefore $\frac{\frac{5}{4} \times 8\frac{1}{2}}{5\frac{1}{2}}$ and the distance $\frac{\frac{5}{4} \times 8\frac{1}{2} \times 14}{5\frac{1}{2}} = 27$ miles 80 yards.

28. $8l.$ $10s.$ $1\frac{2}{3}d.$ 29. $\frac{29}{70} \times \frac{35}{19} \times 1728 = 1318\frac{14}{19}$.

30. $\frac{\frac{13}{15} \times \frac{37}{52}}{\frac{2}{15} \times \frac{3}{4}} \times 2\frac{1}{3} \times \frac{5760}{7000} = 11\frac{21}{25}$.

31. $\frac{9}{250} \times \frac{11}{180} \times \frac{2240}{35} \times 7000 = 985\frac{3}{5}$.

82. 1. $354l.$ $3s.$ $4d.$
2. $337l.$ $3s.$ $7\frac{3}{4}d.$
3. $48958l.$ $6s.$ $8d.$
4. $4581l.$ $10s.$
5. $16098l.$ $3s.$
6. $1737l.$ $14s.$ $7d.$
7. $46169l.$ $12s.$
8. $36910l.$ $8s.$
9. $27845l.$ $5s.$ $6d.$
10. $40257l.$
11. $344l.$ $13s.$ $4d.$
12. $966l.$
13. $42l.$
14. $226l.$ $17s.$ $6d.$

PAGE

82. 15. 80*l.* 17*s.*

16. 37 tons 15 cwt. 0 qrs. 4 lbs.

17. 7352 tons 19 cwt. 1 qr. 20 lbs.

18. 2723*l.* 8*s.* 9*d.* 19. 562500*l.*

20. 14617*l.* 2*s.* 9*d.*

21. Weight, 1 ton 1 cwt. 3 qrs. 21 lbs. 10½ oz.; cost, 35*l.* 16*s.* 9¾*d.*

22. Weight, 1336 tons 10 cwt., cost, 11805*l.* 15*s.*

23. Wrought iron costs 206*l.* 5*s.* more.

83. 24. The chain cable weighs 5 tons 17 cwt. 3 qrs. 12 lbs. more than the wire-rope, but costs 24*l.* 7*s.* 6*d.* less.

25. 4705*l.* 7*s.* 4¾*d.*

26. 4533*l.* 1*s.* 1*d.* There was an oversight in setting this question, as 19 chains make 1 furlong and 9 chains.

27. 951107*l.* 12*s.* 10½*d.*

28.

£	*s.*	*d.*
1	4	8½
	11	7
	7	1
	18	8
	5	8½
£3	7	9

29.

£	*s.*	*d.*
1179	8	5½
7	1	5½
23	7	6
3	11	5
£1213	8	10

30.

£	*s.*	*d.*
33	10	10
12	5	5
15	1	0
9	12	0
4	11	6
2	15	0
	15	0
£78	10	9

31.

£	*s.*
6581	5
2515	10
210	0
263	10
1764	0
661	10
£11995	15

PAGE

32.

£	s.	d.
750	0	0
950	0	0
2160	9	$4\frac{1}{2}$
64707	0	$7\frac{1}{2}$
6725	5	0
48865	7	6
2305	13	9
6439	4	$4\frac{1}{2}$
1245	1	$10\frac{1}{2}$
551	5	0
£134699	7	6

33.

	£	s.	d.
	1211	16	$6\frac{1}{2}$
	297	5	$7\frac{1}{2}$
	186	4	$11\frac{1}{2}$
	42	15	6
	6	1	6
Total cost	£1744	4	$1\frac{1}{2}$

Net weight, 21 tons 17 cwt. 2 qrs. 22 lbs.

Net cost per ton, 79*l.* 14*s.*

1.

93. (1) 72 = 2 × 2 × 2 × 3 × 3.
Divisors, 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36.

(2) 2240 = 2 . 2 . 2 . 2 . 2 . 2 . 5 . 7.
Divisors, 1, 2, 4, 8, 16, 32, 64,
5, 10, 20, 40, 80, 160, 320,
7, 14, 28, 56, 112, 224, 448,
35, 70, 140, 280, 560, 1120.

(3) 5760 = 2 . 2 . 2 . 2 . 2 . 2 . 2 . 3 . 3 . 5.
Divisors, 1, 2, 4, 8, 16, 32, 64, 128,
3, 6, 12, 24, 48, 96, 192, 384,
9, 18, 36, 72, 144, 288, 576, 1152,
5, 10, 20, 40, 80, 160, 320, 640,
15, 30, 60, 120, 240, 480, 960, 1920,
45, 90, 180, 360, 720, 1440, 2880.

(4) 5670 = 2 . 3 . 3 . 3 . 3 . 5 . 7.
Divisors, 1, 3, 9, 27, 81, 2, 6, 18, 54, 162,
5, 15, 45, 135, 405, 10, 30, 90, 270, 810,
7, 21, 63, 189, 567, 14, 42, 126, 378, 1134,
35, 105, 315, 945, 2835, 70, 210, 630, 1890.

PAGE

93. (5) 2646 = 2 · 3 · 3 · 3 · 7 · 7.
Divisors, 1, 3, 9, 27, 2, 6, 18, 54,
7, 21, 63, 189, 14, 42, 126, 378.

(6) 77077 = 7 · 7 · 11 · 11 · 13.
Divisors, 1, 7, 11, 13, 49, 121,
77, 539, 847, 5929,
91, 143, 637, 1001,
1573, 7007, 11011.

2.

(1) 1, 2, 6, 9, 10, 15, 18, 30, 45, 90.
(2) 1, 2, 3, 4, 6, 8, 9, 12, 24, 36, 72.
(3) 1, 2, 7, 14.
(4) 1, 2, 13, 26.
(5) 1, 7, 11, 77.
(6) 1, 2, 4.

3.

The factors are 3^2, $\frac{27}{2}$, $\frac{256}{625}$, 101, $\frac{1}{141}$, $\frac{986}{301}$.

Examples, 4, 6, 7.

		G. C. M.	Quotient	L. C. M.
94.	(1)	8	$\frac{4}{7}$	224
	(2)	24	$\frac{5}{3}$	360
	(3)	9	$\frac{63}{64}$	36288
	(4)	11	$\frac{91}{121}$	121121
	(5)	27	$\frac{27}{37}$	26973
	(6)	16	$\frac{81}{256}$	331776
	(7)	37	$\frac{101}{148}$	553076
	(8)	1	$\frac{4343}{121}$	525503

5 and 6.

	G. C. M.	Quotient		G. C. M.	Quotient
(1)	1259.	$\frac{2003}{28807}$.	(2)	5869.	$\frac{8221}{8269}$.
(3)	7.	$\frac{251683}{249709}$.	(4)	323.	$\frac{2603}{6211}$.
(5)	19.	$\frac{128349}{82211}$.	(6)	2.	$\frac{859223}{2219809}$
(7)	3.		(8)	1.	

PAGE

8.

		G. C. M.	L. C. M.		G. C. M.	L. C. M.
94.	(1)	25.	3000.	(2)	2.	1728.
	(3)	6.	1296.	(4)	30.	900.
	(5)	49.	49098049.	(6)	76201.	2549004064039.

9.

(1) $\frac{1145}{1828824}$. (2) $\frac{109}{809608}$.
(3) $\frac{185211}{2402969}$. (4) $\frac{60872}{7036796361}$.

95. (5) $\frac{266914334055801}{1013281361769600}$. (6) $\frac{98053}{496125}$.
(7) $\frac{444257}{362880}$. (8) $\frac{314513}{362880}$.

10.

(1) $\frac{79}{83}$. (2) $\frac{181}{421}$. (3) $\frac{439}{76201}$. (4) $\frac{2003}{28807}$.

11.

(1) 0. (2) 1. (3) $\frac{1168}{815}$.
(4) $\frac{2129}{1511}$. (5) $\frac{21}{13}$. (6) $\frac{2717}{2472}$.

12.

96. (1) $\frac{1}{8}$. (2) $\frac{6}{121}$. (3) $\frac{3}{2}$.
(4) $\frac{25}{32}$. (5) $\frac{6}{5}$. (6) $\frac{77}{40}$.
(7) $\frac{126}{6875}$. (8) $\frac{907}{983}$. (9) $\frac{927}{997}$.
(10) $\frac{887}{877}$. (11) $\frac{2530775}{120862}$ cwt. (12) $\frac{6509767}{66317440}$ tons.

13.

(1) They are together at 12 o'clock, and cover one the other 11 times in 12 hours at intervals of $65\frac{5}{11}$ minutes. The rest is easy.

(2) Work per hour of both, $\frac{1}{7} + \frac{1}{13} = \frac{20}{91}$ of the reser voir. Hence the reservoir will be drained in $\frac{91}{20}$ hours = 4 hours 33 minutes.

1.

119. (1) $\frac{1}{2}$. (2) $\frac{1}{40}$. (3) $\frac{5}{8}$.
(4) $\frac{27}{4}$. (5) $\frac{1}{3125}$. (6) $\frac{1}{9765625}$.
(7) $\frac{8}{3125}$. (8) $\frac{9}{15625000000}$. (9) $\frac{3}{25}$.

PAGE

119. (10) $\frac{27}{2000}$. (11) $\frac{3}{100000}$. (12) $\frac{22}{10}$.

(13) $\frac{72}{125}$. (14) $\frac{42}{25}$. (15) $\frac{128}{125}$.

(16) $\frac{9241}{10000}$. (17) $\frac{6709}{100}$. (18) $\frac{6693}{1000000}$.

(19) $\frac{2064919}{100000000}$. (20) $\frac{}{1525878906250000}$

2.

(1) 546·1639

(2) 333·3333

(3) 773·7371.

(4) 281·47345

(5) 1451·75609

(6) 111·11111

(7) 3497·57495.

3.

(1) 272·732875.

(2) + 41 7011

120. (3) + 51 208

(4) − 4541.

(5) − 18 65969.

(6) + 0

(7) 346·528185.

4

120. (1) 10. (2) 10

(3) 1190 (4) 11 9

(5) ·0119. (6) ·00119.

(7) ·0999989844 (8) ·001

(9) ·224712 (10) ·00000100000010467.

5.

(1) ·53125. (2) ·0304.

(3) ·1220703125 (4) ·0004096

(5) ·06109619140625. (6) 366·2109375.

(7) 1·8192. (8) ·0009536743164062 5.

(9) ·000000000000033554432. (10) 32000000.

(11) 1000000000000000. (12) ·004096.

6.

(1) ·0882352941. (2) 67·03704.

(3) ·0172566372.

(4) ·000099900099900099900099 9.

(5) ·00090009000900090009000 90.

121. (6) ·000000123456790.

PAGE

121. (7) 859·40. (8) ·001164.
(9) 126·325278. (10) ·007916.
(11) 162583·0359253. (12) 615·070320.

7.

(1) ·3 (2) $\cdot 14285\dot{7}$. (3) $\cdot\dot{7}14285$.
(4) ·4. (5) $\cdot\dot{5}$. (6) $\cdot\dot{1}$.
(7) ·09. (8) $\cdot 6\dot{3}$. (9) $\cdot\dot{9}\dot{0}$.
(10) 076923. (11) ·153846. (12) $\cdot\dot{1}506849\dot{3}$.
(13) ·042735. (14) ·639501.
(15) $\cdot 08\dot{3}$. The period is $\dot{3}$. (16) $\cdot\dot{0}0010999\dot{8}9$.
(17) ·0131211. (18) $\cdot\dot{0}0171699828\dot{3}$.

8.

(1) $\frac{32}{33}$. (2) $\frac{1}{7}$. (3) $\frac{3}{7}$. (4) $\frac{1}{39}$. (5) $\frac{2}{39}$. (6) $\frac{38}{39}$.
(7) $\frac{37}{39}$. (8) $\frac{1}{99}$. (9) $83\frac{83}{99}$. (10) $\frac{8}{9}$. (11) $\frac{32}{137}$.
(12) $\frac{1}{1649}$. (13) $\frac{1}{9901}$. (14) $\frac{1}{17}$. (15) $\frac{1}{29}$.

9.

122 (1) $\frac{19}{60}$. (2) $\frac{3}{22}$. (3) $\frac{9}{55}$. (4) $\frac{953}{1650}$. (5) $\frac{16}{27}$.
(6) $\frac{280}{33}$. (7) $\frac{32}{35}$. (8) $\frac{285}{407}$. (9) $\frac{122}{407}$. (10) $\frac{27587}{36300}$.

10.

(1) 2·7182818. (2) ·3678794. (3) ·8414710.
(4) ·4596977. (5) ·5222374. (6) ·4920587.

13.

123. (1) ·125, ·05, ·0125, ·00416. ·0010416.
(2) ·05, ·0125, ·0004464286, ·0000279018 of a ton.
·25, ·008928571. ·000558036 of a cwt.
·03571429. ·002232143 of a quarter.
1 oz. = ·0625 of a lb.

124. (6) 57·29578.

14.

(1) ·875. (2) ·833. (3) ·474.
(4) 3·999. (5) 5·58. (6) 4·69.

PAGE

15.

124. (1) 124·2126 feet. (2) 6064·19 mètres
(3) 18984213 grammes. (4) 55523 grains.
(5) 101·929. (6) 3657·42.
(7) 395 a. 1 r. 35⅓ p. (8) 61·0269.
(9) 4·536. (10) 101 ft. 833 in.

16.

(1) ·7452288. (2) 13·95625 oz. = 1·16302 pounds
(3) ·31831 of the semi-circumference and equal to radius
(4) 4·543818 sq. m. = 2908·04375 acres.
(5) 15·2802 tons.
125. (6) 10·3764. (7) 21·8685 lbs. (8) 68·705

17.

(1) 62 cwt. 3 qrs. 9 lbs. 2·68 oz.
112594·68 oz., or 49260173 grains.
(2) 57° 17′ 44·8″ = 206264·8″.
(3) 158 days 12 hours 21 minutes 44 seconds.
(4) 369·416415917 acres. 16091779·0773 sq. ft
(5) 17 cubic feet 135·5985 cubic inches.
(6) 106·4348 gallons.

18.

(1) 1·48819. (2) ·671957. (3) 16·019
(4) ·20481. (5) 3·95464. (6) ·005454.
(7) 3·883. (8) 2·9167.
126. (9) 62·321 lbs. 35·943 cubic feet. (10) 10·2694 lbs.
(11) ·76818. (12) ·246857.

1.

131. (1) 12½. (2) 26⅔. (3) $\frac{25}{24}$. (4) $\frac{77}{240}$. (5) ·655976.
(6) 35. (7) 1·54741. (8) 1·34094.

2.

Average price, 9·604*d*.
Average sale, 1104.
Average cash business, 44*l*. 3*s*. 8*d*.

PAGE

131. 3.

Increase per cent.	Decrease per cent.
—	6·054
—	·978
13·204	—
—	3·006
·761	—
—	10·146
—	5·530
—	6·835
2·594	—

132. 4. In this case we must work back to the total number who obtained good or fair marks, by means of the number in the second column, in order to obtain the gross percentages :

	Good	Fair
Mean of Percentages .	21·58	65·59
Gross Percentage . .	25·14	64·5

5. The mean of these distances is 2·655. Taking this as the mean distance, the area is 265·5 square yards. But this is a very rough process. By a more exact method, which is explained in page 214, we get 274·3 as the area, and therefore 2·743 as the mean breadth.

6.

133. (1) 3·75. (2) 2·92. (3) 2·5. (4) 7·5.
(5) 33·3. (6) 37·5. (7) 83·3. (8) 1·5625.

7.

In this case we must first find the actual numbers sick in each garrison, and add them. The answer is :

PAGE

133. France, 11·21; Italy, 13·54; India, 12·39; Gross percentage, 12·23.

8. Artillery and cavalry, 39·25; infantry, 33·44.

1.

139. (1) 13*l.* 2*s.* 6*d.* (2) 865*l.* 3*s.* 4*d.*

(3) 247*l.* 10*s.* (4) 8*l.* 1*s.* 11½*d.*

(5) 307*l.* 6*s.* 4¾*d.* (6) 8577*l.* 3*s.* 1¼*d.*

2.

(1) 2857*l.* 2*s.* 10*d.* (2) 5*l.* 14*s.* 3*d.*

(3) 4 years 162 days. (4) 38095*l.* 4*s.* 9*d.*

(5) 85714*l.* 5*s.* [illegible] (6) 13 per cent. per ann

(7) The actual loan was 18*l.*, and the time may be *averaged* as 5 months, part being repaid earlier, and part later; 2*l.* interest for this is at the rate of 26*l.* 13*s.* 4*d.* per cent. per annum.

(8) 8*l.* 6*s.* 8*d.* per cent., allowing no discount for earlier return of part of capital.

3.

140. (1) 3*l.* 7*s.* 5*d.* (2) 61*l.* 7*s.* 8*d.*

(3) 4414*l.* 5*s.* 9*d.* (4) 1*l.* 10*s.* 5*d.* per cent per an

(5) The interest at 7½ on 75*l.* is $\frac{45}{8}$, and this exceeds the 4*l.* a year interest on the stock by $\frac{45}{8} - 4 = \frac{13}{8}$. The time will be got by dividing this into $95 - 75 = 20$, which gives $\frac{160}{13} = 12$ years 112 days.

(6) I buy with 92*l.* and get 3*l.* a year. The investment pays $\frac{85}{100} \times 5 = 4\frac{1}{4}$*l.* a year, and I have therefore to recoup the loss of capital $(92 - 85 = 7)$ at the rate of $(4\frac{1}{4} - 3 =)$ $\frac{5}{4}$*l.* a year. The number of years is thus $\frac{4}{5}$ of $7 = 5$ years 219 days.

4.

(2) 130*l.* 17*s.* 2½*d.* (3) 2893*l.* 7*s.* 1*d.*

(4) 2116*l.* 15*s.* 7*d.* (5) 1875*l.* 1*s.* 8*d.*

141. (6) 5717*l.* 10*s.* 8*d.*

PAGE 5.

141. (1) 4,571,308. (2) 773,358.

(3) The annual increase is 1·5 per cent. The increase will therefore be 2762.

146. 1. A 18*l.*; B 12*l.*; C 60*l.* 2. B $\frac{1}{7}$*l.*; C 1$\frac{2}{7}$*l.*

3. $\frac{920}{437}$ (or $\frac{40}{19}$) lbs. lead, and $\frac{31180}{437}$ lbs. tin.

4. I buy it for 239*l.* 8*s.*, and sell it for 324*l.* 12*s.* 6*d.* The profit on the outlay is therefore 35·6 per cent.

147. 5. The cost price must evidently be 25*s.* a thousand for 30*s.* to give 20 per cent. profit. From this must be subtracted dues, labour, and coal, together 15*s.* This leaves 10*s.* per 1000 to meet the fixed charges of 520*l.* a year or 10*l.* a week. Hence 20,000 bricks a week must be made.

6. She consumes 120 tons of coal a day. Including wages, therefore, every ton of coal burnt means 13*s.* 8*d.* spent. Every mile of distance between the ports costs 6*s.* 10*d.*, and takes half a ton off her paying cargo. The remainder of the paying cargo must therefore be as many tons as there are farthings in 6*s.* 10*d.*, or 328. The coal must therefore be 2672 tons, and the miles will be 5344.

7. Assuming the consumption duty free to be 10; then at

1*d.* duty	the imports are		9,	yielding	9*d.*
2*d.*	,,	,,	8,	,,	16*d.*
3*d.*	,,	,,	7,	,,	21*d.*
4*d.*	,,	,,	6,	,,	24*d.*
5*d.*	,,	,,	5,	,,	25*d.*
6*d.*	,,	,,	4,	,,	24*d.*

Using whole numbers then, 5*d.* is the highest paying duty. A little easy reasoning shows that this is absolutely true. For if we alter 5*d.* by a small fraction either way (say by ·001), we diminish the revenue. In fact at

PAGE

147. 4·999*d.* duty the imports are 5·001, yielding 24·999999*d.*
5·001*d.* „ „ 4·999 „ 24·999999*d.*
both of which yield less than 25. The reasoning is easily seen to be general.

8. 130·86 imperial gallons.
9. 2·8083 roubles per R. lb.
10. 101*l.* 8*s.* 1*d.* per acre.

148. 11. Multiply the miles per hour by ·447, and the product will be mètres per second.

12. 39·37572. But note that this was the *intended*, not the actual, mètre.

152. (1) 136·61 lbs. (2) 2·62.
(3) ·7282 oz. (4) 2*l.* 16*s.* 8½*d.*
(5) ·2683 of a cubic inch. (6) 1·2.
(7) 7·6056. (8) $\frac{33}{13}$ = 2·54.
(9) Weight, 2·863 oz. Bulk, 2·304 cubic inches. Specific gravity, 2·1475.
(10) ·6 per cent. contraction.

159. (1) 15·887, 6·320, ·003963.
(2) 258·5, 16·068, ·003878.
(3) 254·4, 6·366, ·003931.
(4) 15·921, ·003945.
(5) 16·060, 6·365.
(6) ·02555, with possible error in fifth decimal place.

169. 1. (1) 66. (2) 138. (3) 208. (4) 809.

170. (5) 3057. (6) 3507. (7) 9668. (8) 9999.
(9) 5764801. (10) 9509900499.

2. 43046721, 6561, 81, 9, 3.

3. (1) 9859. (2) 3117·6895452. (3) 985·9.
(4) 311·7689545. (5) 98·59. (6) 31·1768955.
(7) 9·859. (8) 3·1176896. (9) ·9859.
(10) ·3117690. (11) ·09859. (12) ·0311769.

4. (1) 62·2655603. (2) 1·96909902. (3) 75·3856750.
(4) 1·7724538. (5) ·1772454. (6) 2·5066283.

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170. 5. 3733·5. 6. 29667. 7. $1\frac{1}{4}$ in. sq.
8. ·6324555 in. sq. 9. 414. 10. 8·7749644.
11. (1) 84. (2) 306.
(3) 659. (4) 1421.
(5) 7009. (6) 9039.

171. 12. (1) 1·259921. (2) ·310723. (3) 9·579209.
(4) 2·063778. (5) 21·544346. (6) 46·415888.
13. 42 feet. 14. 5·7324.
15. The copper cube, ·4852855 feet a side.
The vessel, 2·060643 feet a side.

181. 1. 544000 foot-pounds.
2. The mean height the water has to be raised is 175 feet.
350 × 18 × 175 = 68744734 foot-pounds.
3. $\frac{2 \times 63{\cdot}6 \times 144 \times 10 \times 25 \times 42}{33000} = 5828\frac{4}{55}$.

There is a little uncertainty in the practice of engineers as to what a stroke is. The above answer assumes each stroke to be single. But if the stroke to and fro be counted as one stroke, the answer will be double that given.

4. The effective H. P. is $\frac{20000 \times 10 \times 11}{550} = 4000$;

the efficiency is $\frac{4000}{5828\frac{4}{55}} = 0{\cdot}686$.

182. 5. Here 15 foot-tons is done in 5 seconds. The work of each jack is therefore $3\frac{3}{4}$ foot-tons, or 3·05 H. P. is done by each, and 5·1 H. P. applied to each.

6. $\frac{5 \times 2240 \times 5 \times 25}{550} = \frac{28000}{11} = 2545$ H. P.

Note that this cannot be done by the action of gravity alone in bringing down the hammer.

7. 10 × 144 × 15 = 21600 foot-pounds.

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182. 8. Height due to velocity $\frac{1440000}{64\ 4} = 22360$ ft Answer

$$\frac{22360 \times 600}{2} = 6708000 \text{ foot-pounds}$$

9. 203 H. P.

10. Velocity due to the height $\sqrt{5280 \times 64\ 4} = \sqrt{340032}$ $= 583\ 1$ feet per second, for the muzzle velocity

Work done by the shot $\frac{5280 \times 5\ 30}{7000} = 399\ 8$ ft -lbs

Work done by the powder $\frac{3998}{7}$ 571 ft -lbs

11 Velocity, 61 6 feet per second
Height due to velocity, 58 9 feet
Work, 10013 foot tons

12. 8035 foot-tons

13 Height due to velocity, 09777 mètres
Work, 1357944 kilogrammètres per second
= 18106 H P

14. 60 cubic feet per second passes over the fall of 40 feet, but part of this work is lost by the tail-race having 4 feet per second more velocity than the head-race The effective height is therefore 39·75 feet, and the

$$\text{H. P.} = \frac{60 \times 39\ 75 \times 62\ 5 \times 0\ 83}{550} - 225$$

15. The number of pounds per second for each foot of sectional area is $\frac{10 \times 5280 \times 62\ 5}{3600}$ Hence the lost work per foot of area is

$$\frac{10 \times 5280 \times 62·5 \times 180}{3600 \times 550} = 300 \text{ H. P}$$

183. 16. The height or head due to 50 miles an hour is 84 2. The loss of head is therefore 35·8 feet.

17. The velocity of issue on the level is 196·56 feet,

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183. and that due to a height of 120 feet is 87·91 per second. The delivery is therefore $\frac{108·65 \times 1815}{196·56}$ = 1009 gallons per minute. The area of nozzle is $\frac{1815 \times 277·274}{60 \times 196·56 \times 12} = 3·56$ square inches.

18. The quantity of water in motion is 26,400,000 cubic feet = 1,650,000,000 lbs. The height due to the velocity is 1·5528 ft. Answer, 2,562,120,000 foot-pounds.

19. 776,400 H. P.

20. The difference of height is 6 feet. The difference of velocity due to this is 19·65 feet per second.

21 The total weight to be moved is $2\frac{1}{4}$ lbs., while only $\frac{1}{4}$ lb. is free for gravity to act upon. Hence the work done (which is proportional to the height) will only be one-ninth of that due to gravity on the whole mass falling freely. Therefore the space traversed in 3 seconds will be 16·1 feet, and the acquired velocity 32·2 feet per second.

22. $\frac{1 \text{ ton}}{144} = 15\frac{1}{2}$ lbs. 23. $\frac{540}{12}$ ft.-tns. = 45 ft.-tns.

24. The moment of the load about the pivot is 60 foot-tons, and of the crane 25. The difference of 35 foot-tons must be balanced by the engine and boiler, the centre of which must therefore be 7 feet behind the pivot.

MENSURATION.

PAGE

189. 1. Diameter, 18·255; length, 33·335 inches.
2. 30 gallons 3 pints.

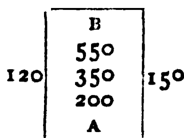
190. 3. 1·147 inches. 4. 6075 tons.
5. $31\frac{1}{2}$ cwt. 6. 2·8284 gallons.

196. 1. 12·649, 12·369, 5 and 13 feet.
2. 24 feet. 3. 362·0387 feet.
4. 493·77 cubic mètres.
5. 17 yards.
6. 105 and 145 yards.
7. 85° 56′ 37·2″. 8. 123·68. 9. 25·447 inches.
10. 23·873 inches. 11. 6086·5 feet.
12. When the diameter is exactly 1 inch, the rule gives ·99745. The rule is therefore about $\frac{1}{4}$ per cent. in defect.

1.

207. (1) 22500 sq. yds. (2) 35000 sq. yds. (3) 74250 sq. ft.
(4) 13950 sq. yds. (5) 18775 sq. yds.
(6) Because the area is independent of the position of the stations.

Field-book.



2. (1) 195·53 yards; 15996 square yards.
(2) 92928 square yards.

208. (3) 24 feet. (4) 13 feet.
(5) 200·31 sq. yds. (6) 1414·2 sq. ft.

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208. (7) $\frac{15 \times 8}{17} = 7\cdot0588$ yds. (8) $\frac{125}{9} = 13\cdot889$ in.

3. (1) Area, 69 sq. in.; BC = 6·7082; CD = 4·9027 in., &c.
(2) 1·125 sq. ft. (3) 16 inches.

209. (1) 654 sq. in. (5) 8489·5. (6) 9375.

210. 4. (1) (*a*) ·141592 in defect. (*b*) ·32251 in excess.
(*c*) ·090459 in excess. (*d*) ·013108 in excess.

(2) Taking the side of the octagon as unity, that of the square is $1 + \sqrt{2}$. The area of the square is therefore $3 + 2\sqrt{2}$, and the four corners are equal to the square on the side of the octagon. The area of the octagon is therefore $2 + 2\sqrt{2} = 4\cdot828427$ square units.

(3) (*a*) − ·135879. (4) (*a*) − ·035398.
(*b*) + ·080627. (*b*) + ·018450.
(*c*) − ·027026. (*c*) − ·008474.
(*d*) + ·008459. (*d*) + ·000501.

(5) 9 acres 9 chains 368 sq. yds. (6) 1766 yds.

217 1. By the trapezoidal rule 20·76, and by Simpson's 20·784 square inches.

218. 5. By the trapezoidal rule, omitting half the ordinates, the value is 20·688; and by Simpson's rule 20·8 exactly. This coincidence is accidental, the ordinates being only given approximately.

6. By the trapezoidal rule, 2·435 square inches.

220. 1. 78·5398 square inches. 2. 38·4845 square inches.
3. 1·784 in. 4. 58·012 square inches.
5. 3·545 miles. 7. 2 r. 13 p. 14 yards.
8. 2·477 inches. 9. 8 inches.
10. 5·1614 square inches. 11. ·0002 in excess.

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223. 1. 3·534 square feet. 2. 9·512 inches.
3. 127° 19′ 26″. 4. 4·5664 square inches.
5. 3·2611 square inches. 6. 11·27 square inches.
7. 5708 square feet. 8. 3056 square feet.

227. 1. $\frac{3}{2}\pi = 4·7124$ sq. in. 2. $\frac{1}{2}\pi = 1·5708$ sq. mile.
3. $\frac{27}{8}\pi = 10·603$ ft. 4. $ab \times \left(\frac{\pi}{4} - \frac{1}{2}\right) = ab \times ·2854$.
6. $\frac{100}{8\pi} = 3·97$. 7. $\sqrt{\frac{24}{2} \times \frac{36}{2}} = 14·697$.
8. $\frac{21}{2}\pi = 32·9867$.

232. 1. 28. 2. 636 cub. ft. 298 cub. in.
3. 2819 gallons. 4. 1092 cubic feet.
5. 357 cubic feet. 6. 7·67 inches.
7. 7·068 inches. 8. 23 lbs.
9. 7987 lbs. = 3 tons 11 cwt. 1 qr. 7 lbs.
10. $160 \times 25 \times \frac{\pi}{4}$ cubic feet to be raised a mean height of 120 feet, making 10519 foot-tons.
11. $\frac{2 \times 42 \times 42 \times \pi \times 28 \times 4 \times 30}{33000} = 1128·5$
12. 148·7.

234. 1. 7·06 inches, measured on the side of the cylinder.
2. 4·7124 cubic inches; 3·175 inches high.
3. 62720. 4. 100. 5. 13416.
6. 13·9 cubic inches. 7. 55 lbs.

235. 8. 209.

238. 1. 179·594 cubic inches. 2. 4·045 inches.

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238 3. 1·21 square ft. 4. ·7523 cubic ft.
5. 263 lbs. 5 oz. iron : 11 lbs. 11 oz. powder.
6. 1,082,880,000,000,000,000,000.
7. 25,700,000,000,000,000,000.
8. 113·1 cubic feet. 9. 1·51 gallons.
10. ·28 of a gallon. 11. $\frac{5}{27}$.

239. 12. ·53 of a gallon.

241. 1. 37·7 and 51·84 square inches.
2. 6·2432 square feet and 1·5708 cubic ft.
3. 8·5046 square feet : because the areas of the end vary with the inclination.
4. 162 square inches.
5. 10 tons 6 cwt. 0 qrs. 12 lbs. 6. 3*s.* 7½*d.*

243 1. 942·48 square feet. 2. 1570·8 square inches.
3. 133·5 square inches. 4. 835·67 square inches.
5. 202 lbs. 5 oz.
6. 14·71 inches.
7. The radius of the base is half the slant side. Section of the cone through axis is an equilateral triangle, and angle of slope 60°.

246. 1. 452·39 square inches, or 3·1416 square feet.
2. 218 lbs. 3. 3·927 square feet.
5. 1·118 ft. radius. 6. 155,450,000,000,000 tons.
7. ·2755 of an inch. 8. 17·81.

248. 1. $480\,\pi^2 = 4737$ square feet.
2. Volume $= \frac{1}{2}\pi\sqrt{3} = 2{\cdot}72$ cubic inches.
Area $= 6\,\pi = 18{\cdot}8496$ square inches.
3. Taking the radius as r, the distance of the centre of gravity of the semicircle is $\frac{4r}{3\pi}$ from the centre of the circle ; and for the arc $\frac{2r}{\pi}$.

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248. 4. *First*, for the volume—

The centre of gravity is distant $6 + \frac{12}{3\pi}$ inches from the axis. Its path is therefore $12\pi + 8$ Multiplying this by its area $\frac{9\pi}{2}$ square inches, we have for the volume

$$54\pi^2 + 36\pi = 646 \text{ cubic inches.}$$

Secondly, for the area—

The centre of gravity is distant $6 + \frac{6}{\pi}$ inches from the area. Its path is therefore $12(\pi + 1)$. Multiplying this by its length, 6π, we have for the surface swept out

$$72\pi^2 + 72\pi = 936{\cdot}6 \text{ square inches.}$$

This does not include the cylindrical surface swept out by the diameter.

5. The volume is evidently the same as if the centre revolved in a circle instead of moving on a screw. Hence it is $5\pi^2$, or $49{\cdot}35$ cubic inches.

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